THE EFFECTIVE EINSTEIN EQUATIONS ON THE BRANE: REVIEW

Topics

(i) The effective equations

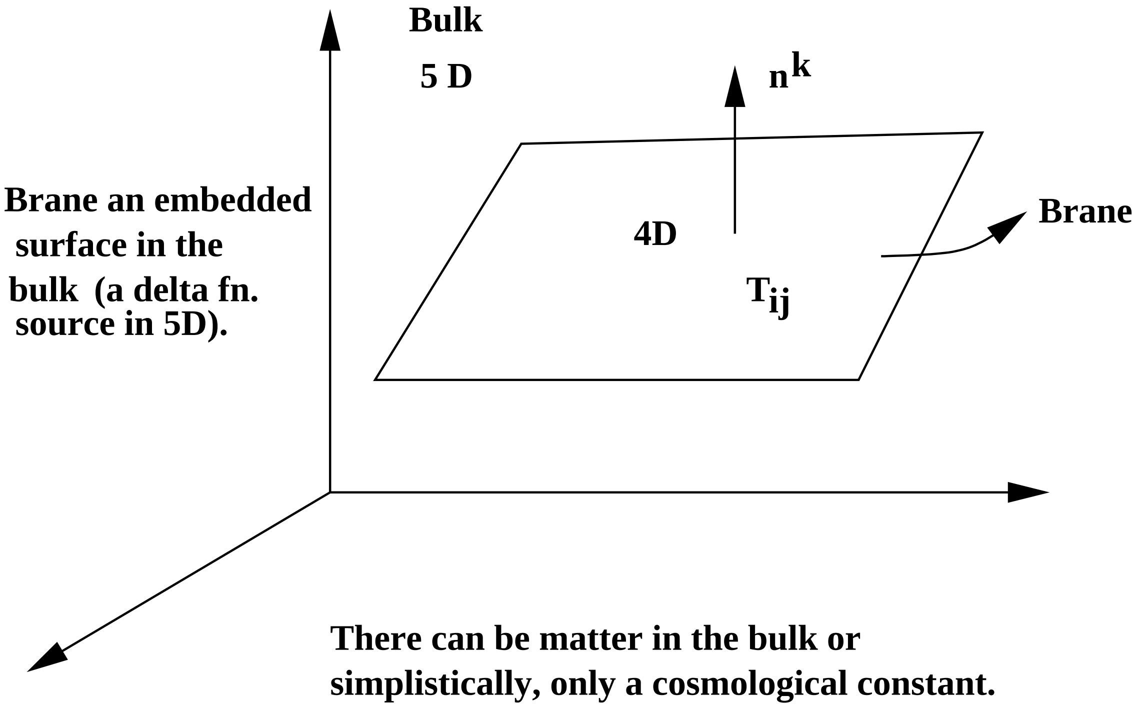
(ii) How to derive them

(iii) Solutions: Cosmology

(iv) Solutions: static, spherisymmetric

(v) Modeling cluster halos

EFFECTIVE EINSTEIN EQUATIONS



$$g_{ab} = h_{ab} + n_a n_b, \; ds^2 = g_{ij}(x^i, \sigma) dx^i dx^j + d\sigma^2$$

- What are the Einstein equations induced on the four dimensional brane from the five dimensional Einstein equations in the bulk?

- Are they the usual Einstein equations or are there any novelties that arise?

THE EQUATIONS

$$G_{ij} = -\Lambda h_{ij} + \kappa^2 T_{ij} + 6\frac{\kappa^2}{\lambda} Q_{ij} - \mathcal{E}_{ij}$$

$$\kappa^2 = \tfrac{1}{6}\lambda\kappa_5^4,\ \Lambda = \tfrac{1}{2}\left[\Lambda_5 + \kappa^2\lambda\right]$$

$$Q_{ij} = \tfrac{1}{12} T T_{ij} - \tfrac{1}{4} T_{ik} T_j^k + \tfrac{1}{24} h_{ij}\left[T_{kl}T^{kl} - T^2\right]$$

$$\mathcal{E}_{ij} = {}^{(5)}C_{abcd} n^a n^c h_i^b h_j^d$$

- **Compare with usual Einstein equations**

$\rightarrow$ h_{ij} is induced metric on the brane
$\rightarrow$ Quadratic in T_{ij} contribution via Q_{ij}
$\rightarrow$ Extra term $\mathcal{E}_{ij}$ with properties:
(i) traceless (ii) no off–brane component

$\rightarrow$ Usual Einstein equation recovered for situations where (i) the bulk Weyl is zero (ii) quadratic contributions small

HOW TO DERIVE THEM?

• **Bulk Field Equations:**

$$^{(5)}G_{ab} = -^{(5)}\Lambda g_{ab}$$

• **Coordinates, normals, induced metric, extrinsic curvature**

$\rightarrow$ Brane located $\sigma = 0$

$\rightarrow$ Brane tension λ, on–brane matter $T_{\mu\nu}$

$\rightarrow$ σ a Gaussian normal coordinate orthogonal to the brane ($\sigma = 0$)

$\rightarrow$ $^{(5)}g_{ab} = h_{ab} + n_a n_b$, n^a unit normal, $d\sigma = n_a dX^a, X^a$ bulk coordinates $\rightarrow$ Extrinsic curvature : $K_{ab} = g_a^{c(5)}\nabla_{(c}n_{b)}$

• **Effective on–brane Einstein equations**

$\rightarrow$Encodes information of brane embedding, boundary conditions

$\rightarrow$ Uses (i) Gauss–Codazzi eqns (ii) projections of 5D tensors into 4D ones (iii) decomposition of Riemann tensor into Ricci and Weyl

THE RELEVANT STEPS

- **Gauss–Codazzi equations**

$$R_{abcd} = {}^{(5)}R_{efgh}g^e_a g^f_b g^g_c g^h_d + K_{ac}K_{db} - K_{ad}K_{cb}$$

$$\nabla_b K^b_a - \nabla_a K = {}^{(5)}R_{bc}n^c g^b_a$$

- **Riemann=Ricci+Weyl**

$$ {}^{(5)}R_{abcd} = {}^{(5)}C_{acbd} + \frac{2}{3}\left({}^{(5)}R_{a[c}g_{d]b} + {}^{(5)}R_{b[c}g_{d]a}\right) - \frac{1}{6}{}^{(5)}R g_{a[c}g_{d]b}$$

- **Use in Gauss eqn to get (on the brane)**

$$G_{ij} = -\frac{1}{2}\Lambda_5 h_{ij} + KK_{ij} - K^k_i K_{jk} - \frac{1}{2}h_{ij}(K^2 - K^{kl}K_{kl}) - \mathcal{E}_{ij}$$

• **How to remove the K terms? Use the Israel junctions conditions**

$$\boxed{\Delta K_{ij} = 2K_{ij} = -8\pi G_5 \left(S_{ij} - \tfrac{1}{3} S h_{ij}\right)}$$

using Z_2 **symmetry**

and $S_{ij} = -\lambda h_{ij} + T_{ij}$ **to get**

$$\boxed{\begin{gathered} G_{ij} = -\Lambda h_{ij} + \kappa^2 T_{ij} + 6\frac{\kappa^2}{\lambda} Q_{ij} - \mathcal{E}_{ij} \\ \kappa^2 = \tfrac{1}{6}\lambda\kappa_5^4,\ \Lambda = \tfrac{1}{2}\left[\Lambda_5 + \kappa^2\lambda\right] \\ Q_{ij} = \tfrac{1}{12} T T_{ij} - \tfrac{1}{4} T_{ik} T_j^k + \tfrac{1}{24} h_{ij}\left[T_{kl}T^{kl} - T^2\right] \\ \mathcal{E}_{ij} = {}^{(5)}C_{abcd} n^a n^c h_i^b h_j^d \end{gathered}}$$

• **Also note conservation law:**

$$6\frac{\kappa^2}{\lambda} \nabla^j Q_{ij} = \nabla^j \mathcal{E}_{ij}$$

GENERAL COMMENTS

$$\boxed{G_{ij} = -\Lambda h_{ij} + \kappa^2 T_{ij} + 6\frac{\kappa^2}{\lambda} Q_{ij} - \mathcal{E}_{ij}}$$

• No effect of $\mathcal{E}_{ij}$ if bulk Weyl is zero, as is the case for the RS model

• Inclusion of bulk matter in the bulk equations leads to an additional term (the $\mathcal{F}_{ij}$ term) dependent on bulk stress energy

• One can construct effective equations in other situations as well:

$\longrightarrow$ for a two brane model: hep-th/0210066

$\longrightarrow$ with a bulk GB term: hep-th/0608166

$\longrightarrow$ with two D-branes: hep-th/0405071

$\longrightarrow$ without Z_2 symmetry: hep-th/0105091

• A major problem with this formalism is that one cannot go back to the bulk–Taylor expansion? So there are many bulk metrics which can give the same brane metric

COSMOLOGICAL SOLUTIONS

• **The Friedmann equations (perfect fluid)**

$$\boxed{\begin{aligned} H^2 &= \frac{\Lambda}{3} - \frac{k}{a^2} + \frac{\kappa^2}{3}\rho + \frac{\kappa^2}{6\lambda}\rho^2 + \frac{\rho_\epsilon}{3} \\ \dot{H} &= \frac{k}{a^2} - \frac{\kappa^2}{2}(\rho + p) - \frac{\kappa^2}{2\lambda}\rho(\rho + p) - \frac{2}{3}\rho_\epsilon \end{aligned}}$$

• **Note the quadratic terms in ρ, p and the terms coming from $\mathcal{E}_{ij}$ (ρ_ϵ).**

• **New solutions?**

$\rightarrow$ **For an expanding universe, it is clear that the quadratic terms will dominate in the early stages of evolution**
$\rightarrow$ **The $\mathcal{E}_{ij}$ term can mimic the dark components of the universe**

SPHERISYMMETRIC, STATIC

- **Effective 'Vacuum' has $\mathcal{E}_{ij}$**

$$R_{ij} = -\mathcal{E}_{ij} \quad ; \quad R = 0 \quad ; \quad \nabla^i \mathcal{E}_{ij} = 0$$

- **Solution similar to Reissner–Nordstrom, except for the possibility of a different sign in the 'charge' term**

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{q}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r} + \frac{q}{r^2}\right)} + r^2 d\Omega^2$$

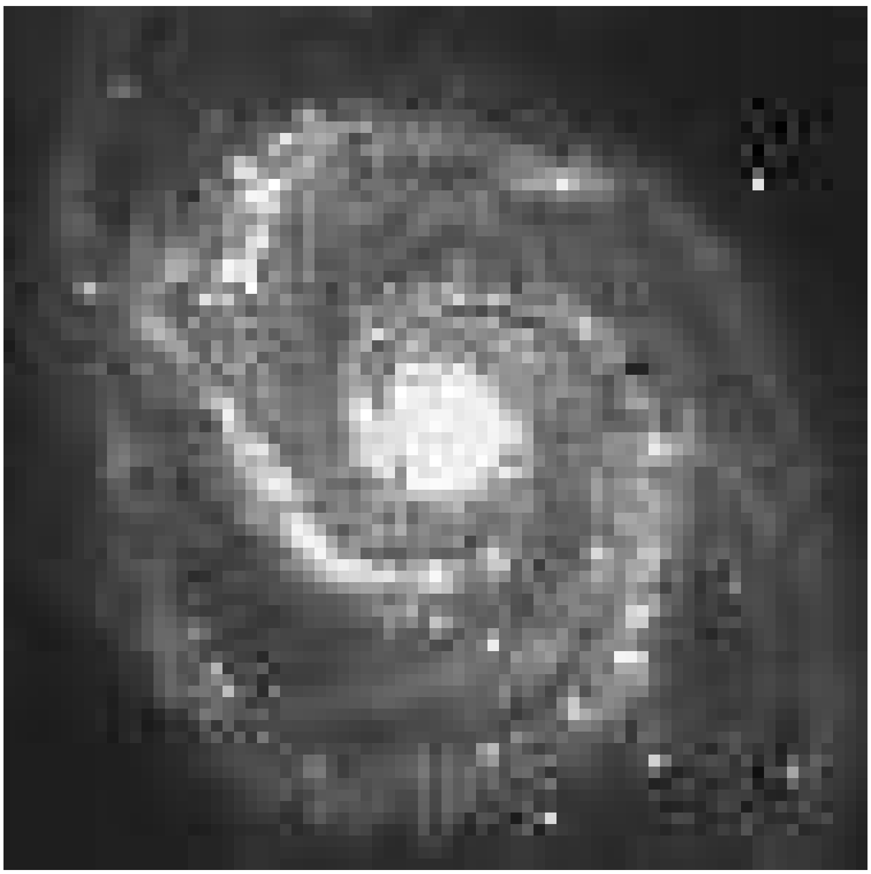
- $q > 0$ **is the RN case,** $q < 0$ **is new. For (**$q < 0$**) there is a single horizon, outside the Schwarzschild horizon which leads to increase in entropy and decrease in temperature.**

- **Note that bulk Weyl tensor effects are the cause**

DARK MATTER AS MODIFIED GRAVITY

Rotation curves of spiral galaxies

• A spiral galaxy is a disk of stars and dust rotating about a central nucleus



The spiral galaxy M31 : also known as the Whirlpool Galaxy

• **Neutral hydrogen clouds are moving along circular geodesics about the galaxy core**

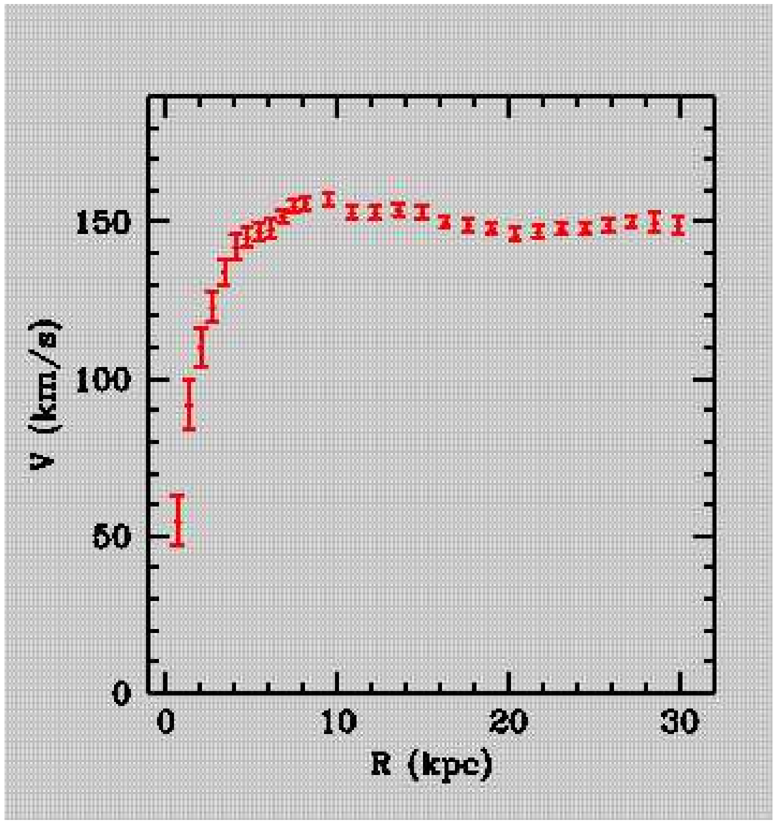
• **Measure Doppler shifts in the 21 cm line of the neutral hydrogen clouds**
$\rightarrow$ Find velocities, v(r) of the neutral hydrogen clouds as a function of distance r from the centre of the galaxy. We expect :

$$\boxed{\frac{GM(r)}{r^2} = \frac{v^2(r)}{r}}$$

$\rightarrow$ Inside the radius $r = R$ containing luminous matter $M(r) \propto r^3$, $v(r) \propto r$
$\rightarrow$ Outside the radius $r = R$ containing luminous matter , $v(r) \propto r^{-\frac{1}{2}}$

• **Puzzle :** $v(r)$ **remains approximately constant beyond** $r = R$ $\rightarrow$ Mass $M(r) \propto 1/r$

• **Typical constant value of v : 200 km/s**



• Thus beyond the region visible to us there is non–luminous or DARK matter, which gravitates

• Dark, non-luminous matter can be probed by gravitational lensing

• Question : Can a modified law of gravity explain the flattening of the rotation curves?

Galaxy clusters and dark matter

• Galaxies are found in gravitationally bound local groups called 'clusters'

→ 'Rich' clusters : several thousand galaxies. 'Poor' ones : (Local Group) 30-50 galaxies

→ Shapes : spherical, flattened, irregular

→ Galactic content : spiral–rich, elliptical–rich

→ Strong radio sources or emit X-rays

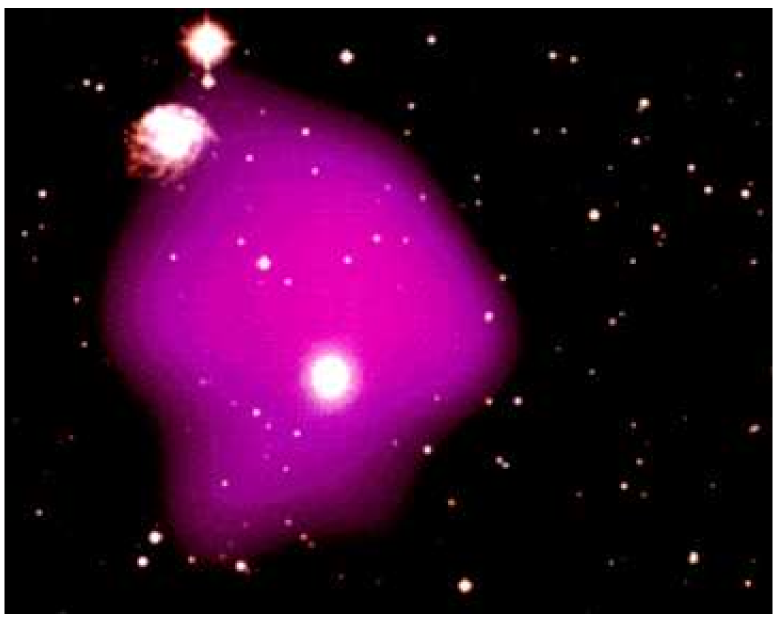
The Virgo cluster : 60 million Ly away, 2500 galaxies, elliptical–rich

X-ray clusters

- **X-ray satellite observations reveal :**

→ Luminous matter in clusters is hot gas at temperature of 10-100 million degrees, radiating X-rays.

→ Amount of hot gas is related to the total X-ray luminosity



X-ray image of cluster superposed on optical image. The pink region shows the hot gas emitting in the X-ray

• **Study how much the gas is being squeezed around by gravity. Estimate the total mass in a cluster**

→ Balance between the dark matter and the pressure of the cluster (related to the X-ray emitting gas).

→ Assuming hydrostatic equilibrium, we can estimate the amount of dark matter.

$$\boxed{\rho_{gas} = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-\frac{3\beta}{2}} \quad ; P_{gas} = \frac{kT}{\mu m_p c^2} \rho_{gas}}$$

→ Conservation law of energy momentum gives the Newtonian potential and hence the total gravitating mass in a cluster

$$\boxed{\frac{d\Phi}{dr} = -\frac{kT}{\mu m_p c^2} \frac{1}{\rho_{gas}} \frac{d\rho_{gas}}{dr}}$$

→ Solve to get Φ and hence the mass

→ The luminous matter is only 10-15 percent

• **Can a modification of the law of gravity explain things without actual dark matter?**

MODELING CLUSTERS AND HALOS

• **Use modified equations** $G_{\mu\nu} + \mathcal{E}_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$

Can the modified effective equations with $\mathcal{E}_{\mu\nu}$ replace the notion of 'dark matter'?

$\rightarrow$ Assume the weak field line element within the galaxy or cluster halo

$$ds^2 = -\left(1 + 2\Phi\right) dt^2 + \left(1 - 2\Phi + 2\Psi\right) \left[dr^2 + r^2 d\Omega^2\right]$$

$\rightarrow$ How to find Φ and Ψ?

• **Φ is obtained from kinematics**

$\rightarrow$ Use geodesic eqn for individual halos

$\rightarrow$ Use the hydrostatic equilibrium criterion for clusters

• **To find Ψ, use Φ above, the modified Einstein eqn and the traceless character of $\mathcal{E}_{\mu\nu}$**

• **Final equation for potentials**

$$\boxed{\begin{array}{c} \nabla^2 (\Phi - 2\Psi) = \frac{4\pi G}{c^2}\rho_{vis} \\ \\ \text{Solution : } \Psi = \frac{1}{2}\Phi - \left(\nabla^2\right)^{-1} \rho_{vis} \end{array}}$$

→ For clusters $\rho_{vis} = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-\frac{3\beta}{2}}$

→ Assume $r >> r_c$ and $\beta = \frac{2}{3}$

→ Using representative values for constants :

$$\boxed{\Phi = \frac{2kT}{\mu m_p c^2} \ln \frac{r}{r_c} \quad ; \quad \Psi = \frac{1}{2}\Phi - \frac{2\pi G \rho_0 r_c^2}{c^2} \ln \frac{r}{r_c}}$$

• **In the usual analysis $\Psi = 0$ and we need dark matter. In the modified theory $\mathcal{E}_{\mu\nu}$ replaces dark matter.**

• **How to distinguish? Lensing calculations reveal → deflection angle is .75 times the Newtonian value !**

CORRECTIONS: NEWTONIAN GRAVITY

- **What are the corrections to Newtonian gravity induced from the bulk?**

- **KK expansion of the graviton modes**

$$ds^2 = e^{-A(z)}\left[\left(\eta_{ij} + h_{ij}(x,z)\right)dx^i d^j + dz^2\right]$$

RS gauge condition: $h^i_i = \partial_i h^i_j = 0$

- **Perturbation equation:**

$$h_{ij} = e^{\frac{3}{4}A}\hat{h}_{ij}\Psi(z), \quad {}^{(4)}\nabla^2 \hat{h}_{ij} = m^2 \hat{h}_{ij}$$

$$-\partial_z^2\Psi + \left(\frac{15}{4}\frac{k^2}{1+k|z|^2} - \frac{3k\delta(z)}{1+k|z|}\right)\Psi = m^2\Psi$$

For large z potential goes as $\frac{1}{z^2}$ (Bessel equation)

In general the potential has the shape of a volcano

Match solutions inside and outside the well to get $\Psi_m(0) = \sqrt{\frac{m}{k}}$

Use $\Psi_m(0)$ **in the formula:**

$$U(r) \sim \frac{GM_1M_2}{r} + \frac{1}{M^3}\int_0^\infty dm \frac{M_1M_2e^{-mr}}{r}\Psi_m^2(0)$$

To get:

$$\boxed{U(r) = \frac{GM_1M_2}{r}\left(1 + \frac{C}{(kr)^2}\right)}$$

- **Note the additional term which could be significant at short distances–short distance tests may detect such terms.**

POSSIBLE OPEN ISSUES

- Understanding the effects of the quadratic stress energy term

- Understanding the effects of bulk matter on the brane equations

- Linking up the Kaluza-Klein modes obtained in a perturbative set-up with the effects of $\mathcal{E}_{ij}$

- Investigating the consequences of the other effective equations.