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Gravitational potential and tidal tensor.

The gravitation acceleration at any point can be expressed as the gradient of a potential $\phi(\vec{x},t)$

$$\boxed{\vec{a} = -\vec{\nabla}\phi(\vec{x},t).}$$

and $\vec{F} = -m\vec{\nabla}\phi(\vec{x},t)$

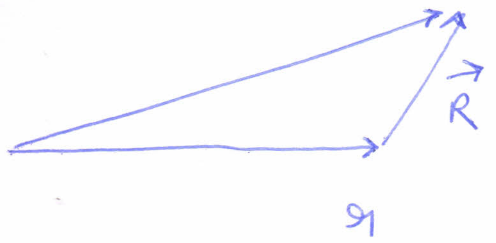
where m is the mass of the test particle located at $\vec{x}$.

$$\phi(\vec{x},t) = -G\int \frac{\rho(\vec{x}',t)\,d^3x'}{|\vec{x}-\vec{x}'|}$$

$$\boxed{\nabla^2\phi(\vec{x},t) = 4\pi G\rho(\vec{x},t).}$$

Poisson equation.

Tidal force.



$R \ll r$.

$$\vec{F}_T(\vec{r},\vec{R},t) = \vec{F}(\vec{r}+\vec{R},t) - \vec{F}(\vec{r},t)$$

$$F_{T_i} = -m\left[\phi_{,i}(\vec{r}+\vec{R},t) - \phi_{,i}(\vec{r},t)\right]$$

$$\simeq -m\,\phi_{,ij} R_j$$ where the derivative is at r.

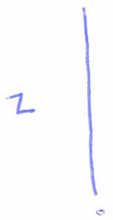
Consider the situation where there is a point mass at $\vec{r}$. This causes an isotropic inwards force which is not tidal. We have to subtract this out.

$$F_T = -m\left[\phi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\phi\right]R_j$$

we define the tidal tensor

$$T_{ij} = -\left[\phi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\phi\right]$$

example along z axis from a mass M.



$$\phi(r) = -\frac{GM}{r}$$

$$\phi_{,i} = GM\frac{r_i}{r^3}$$

$$\phi_{,ij} = GM\frac{r^2\delta_{ij} - 3r_i r_j}{r^5}$$

$$T_{ij} = -\frac{GM}{r^5}\left[r^2\delta_{ij} - 3r_i r_j\right]$$

$$T_{11} = -\frac{GM}{r^3} = T_{22}$$

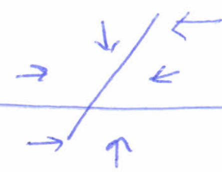
$$T_{33} = \frac{2GM}{r^3}$$

$$\begin{pmatrix} F_{Tx} \\ F_{Ty} \\ F_{Tz} \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}$$

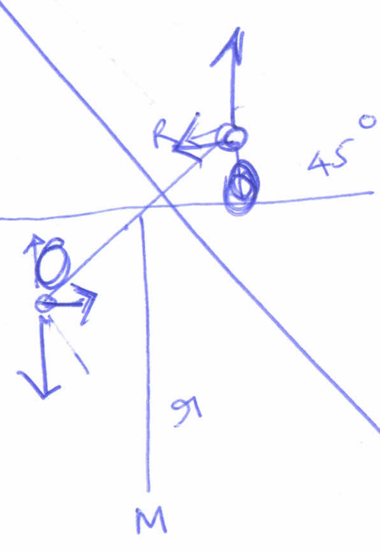
$$F_{Tz} = \frac{2GM}{r^3} R_z \; ; \; F_{Tx} = -\frac{GM}{r^3} R_x$$

$$F_{Ty} = -\frac{GM}{r^3} R_y .$$

As discussed earlier.



Tidal Torque.



Net torque

$$2R \cdot \frac{1}{\sqrt{2}} \frac{GMR}{r^3 \sqrt{2}}$$

$$= \frac{2GMR^2}{r^3}$$

Torque $= \frac{R}{\sqrt{2}} \times \frac{GMR}{r^3 \sqrt{2}} + \frac{2GM}{r^3} \frac{R}{\sqrt{2}} \frac{R}{\sqrt{2}}$

$$= \frac{GMR^2}{r^3} \times \frac{3}{2}$$

in general.

$$\cancel{\tau_i = \epsilon_{ijk} \int \rho(r)(\hat{r} - r_c)(r - r_c)\, d^3r}$$

around center of mass.

$$\tau_i = \epsilon_{ijk} \int (r - r_c)_j \, F_k(\vec{r})\, d^3r$$

$$= \epsilon_{ijk} \int (r - r_c)_j \left[- \rho(r) \left\{ \phi_{,k}(r_c) + \phi_{,kl}(r_c)(r - r_c)_l \right\} \right] d^3r$$

$$= \epsilon_{ijk} \left[\int (r - r_c)_j \, \rho(r)\, d^3r \right] \left[- \phi_{,k}(r_c) \right]$$

$$- \epsilon_{ijk} \, \phi_{,kl}(r_c) \int d^3r \, (r - r_c)_j (r - r_c)_l \, \rho(r)$$

The first integral is zero, definition of C.M.

Second integral.

$$= \epsilon_{ijk} \left[T_{kl} + \frac{1}{3} \delta_{kl} \nabla^2 \phi \right] I_{jl}$$

we have $\epsilon_{ijk} I_{jk} = 0$

I_{jk} is the symmetric moment of inertia tensor.

In a similar way.

$$\tau_i = \epsilon_{ijk} T_{kl} \left[I_{jl} - \frac{\delta_{jl}}{3} I_{mm} \right]$$

traceless part of inertia tensor.

arises due to deviation from spherical symmetry — this is the mass quadrupole tensor.

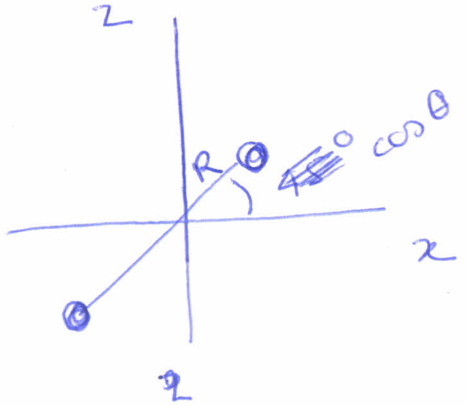
$$Q_{kl} = \int \rho(r) \left[(r - r_c)_l (r - r_c)_k - \frac{\delta_{kl}}{3} (r - r_c)^2 \right] d^3 r.$$

This is the tidal ~~torque~~. arises only if there is a mass quadrupole moment.

Question How does it align the quadrupole relative to the direction of the local gravitation acceleration.

Final

$$\tau_i = \epsilon_{ijk} Q_{jl} T_{kl}$$



$$Q = R^2 \begin{pmatrix} \cos^2\theta & 0 & \cos\theta\sin\theta \\ 0 & 0 & 0 \\ \cos\theta\sin\theta & 0 & \sin^2\theta \end{pmatrix}$$

$$- \frac{2R^2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q_{jk} = 2\cancel{2}R^2 \left[n_j n_k - \frac{\delta_{jk}}{3} \right]$$

$$T_{mj} = -\frac{GM}{r^3} \left[3\hat{z}_m \hat{z}_j - \cancel{3}\,\delta_{mj} \right]$$

$$\epsilon_{imk} T_{mj} Q_{jk} = -\frac{GMR^2}{r^3} \left[\hat{z}_m \hat{n}_j + \frac{\cancel{\hat{z}_m \hat{z}_k}}{3} \times (\hat{z}\cdot\hat{n}) \right.$$

$$\left. \cancel{- 3\hat{n}_m \hat{n}_k + \delta_{mk}} \right] \epsilon_{imk}$$

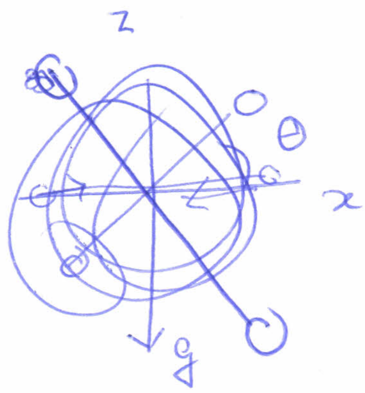
$$= -\frac{6GMR^2}{r^3} \epsilon_{imk} \hat{z}_m \hat{n}_k (\hat{z}\cdot\hat{n})$$

$$= -\frac{6GMR^2}{r^3}\epsilon_{i3k}\hat{n}_k(\hat{n}\cdot\hat{z})$$

Let us take $\hat{n}$ in x-z plane

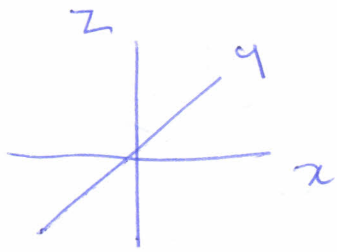
$\hat{n}_1 = \cos\theta$

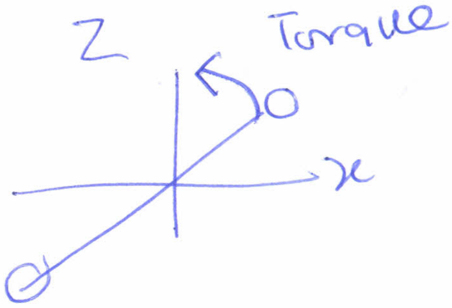
$$\tau_2 = -\frac{2GMR^2}{r^3}\underbrace{\epsilon_{231}}_{=1}\cos\theta\sin\theta$$

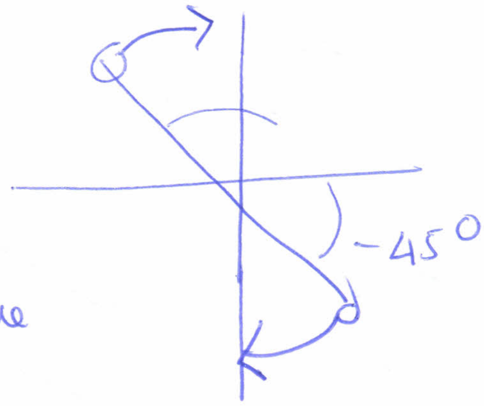


outwards ~~in~~

Torque







align along the direction of $\vec{g}$.

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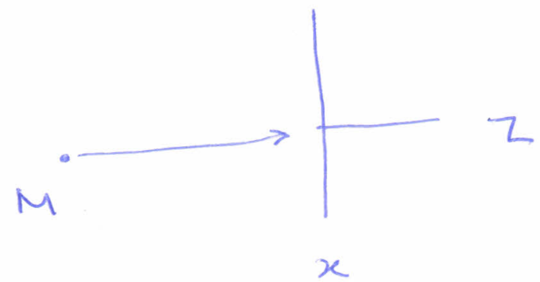
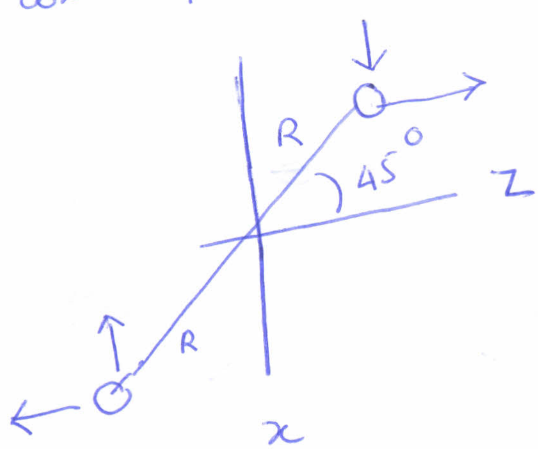
Torque on a dunbell from a mass a distance r away



y axis points inwards



$$(\text{Torque})_y = \frac{2GMm}{r^3} \, 2 \times \frac{R}{\sqrt{2}} \times \frac{R}{\sqrt{2}}$$

$$+ \frac{GM}{r^3} \, 2 \times \frac{R}{\sqrt{2}} \times \frac{R}{\sqrt{2}}$$

$$= 3GMm\frac{R^2}{r^3}.$$

The torque aligns the dumbbell with $\vec{g}$.