Simple harmonic motion (SHM) (Review)

Why shall one study the simple harmonic motion? Straightest answer would be "*its appearance in variety of realistic problems.*" Most of the systems when slightly disturbed from their stable equilibrium positions perform simple harmonic motion. Many of the potentials appearing in physical problems can fairly be approximated by a harmonic oscillator potential. For example inter-atomic/intermolecular forces are well approximated by harmonic oscillator potential. Consider the following intermolecular/inter-atomic potential,

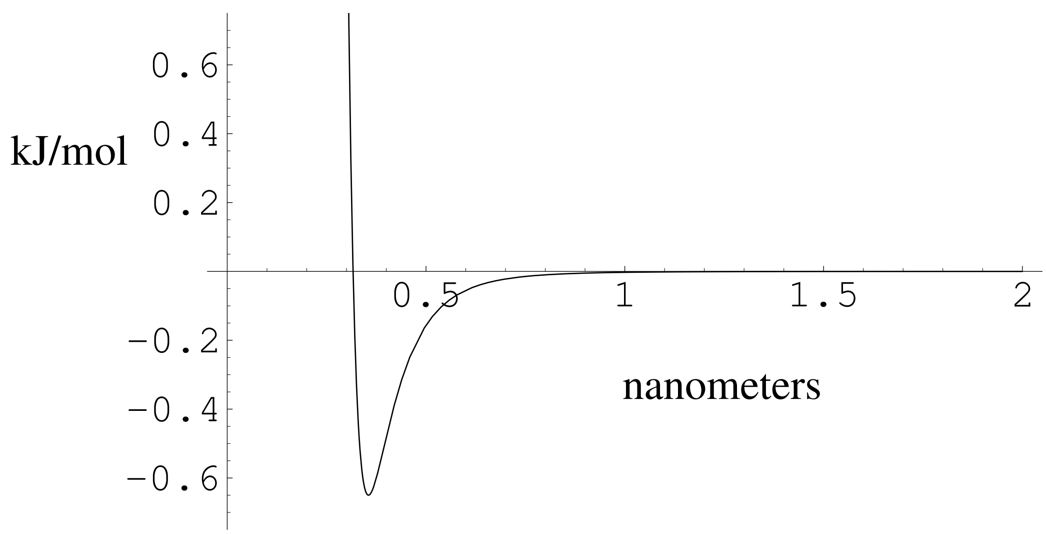

Figure 1: Lennard Jones potential, $V(r) \sim (\frac{a}{r})^{12} - (\frac{a}{r})^6$, for water

the part of the potential around minimum is well approximated by a harmonic oscillator.

Harmonic oscillator potential:

$$V(x) = \frac{1}{2}kx^2 \qquad k = \text{spring constant is a positive quantity} \tag{1}$$

Claim: Arbitrary potentials can be well approximated by harmonic oscillator potential near its minima.

Proof: Consider a potential $V(x)$. Let us assume that it has a minimum at $x = x_0$. Expanding $V(x)$ around $x = x_0$, we obtain,

$$V(x) = V(x_0) + (x - x_0)V'(x_0) + \frac{1}{2!}(x - x_0)^2 V''(x_0) + \cdots, \tag{2}$$

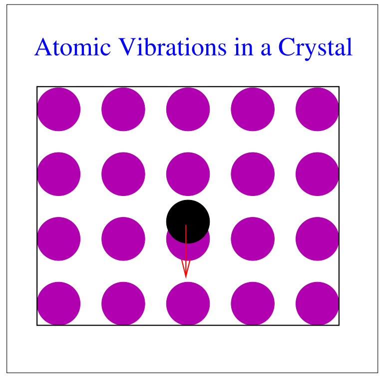

Figure 2: Atomic vibrations in a crystal

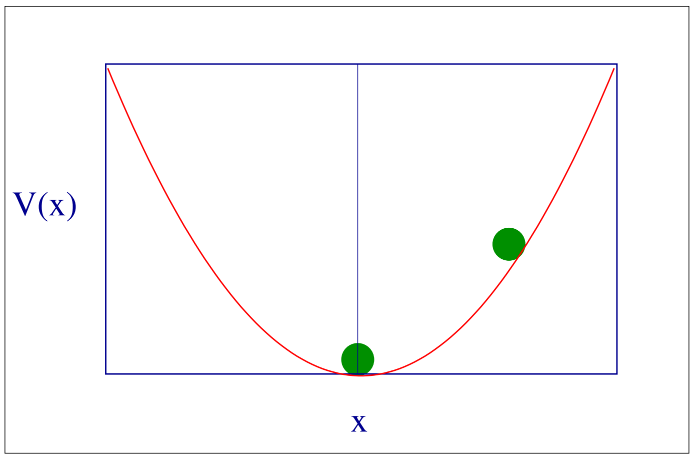

Figure 3: A particle in harmonic oscillator potential

where a prime denotes a derivative with respect to x. In (2) $V^{'}(x_0)$ is zero and $V^{''}(x_0)$ is positive(in most cases) since x_0 is a minimum of $V(x)$.

Problem: Find explicit examples of potentials where the above is not true. i.e. the potential cannot be approximated by a harmonic oscillator around its minimum.

The negative derivative of the potential is the force.

Hooke's law:

$$F = -kx \tag{3}$$

The force is proportional to the displacement.

A particle in harmonic oscillator potential:

$$\begin{aligned} m\ddot{x} + kx &= 0, \\ \ddot{x} + \omega_0^2 x &= 0, \end{aligned} \tag{4}$$

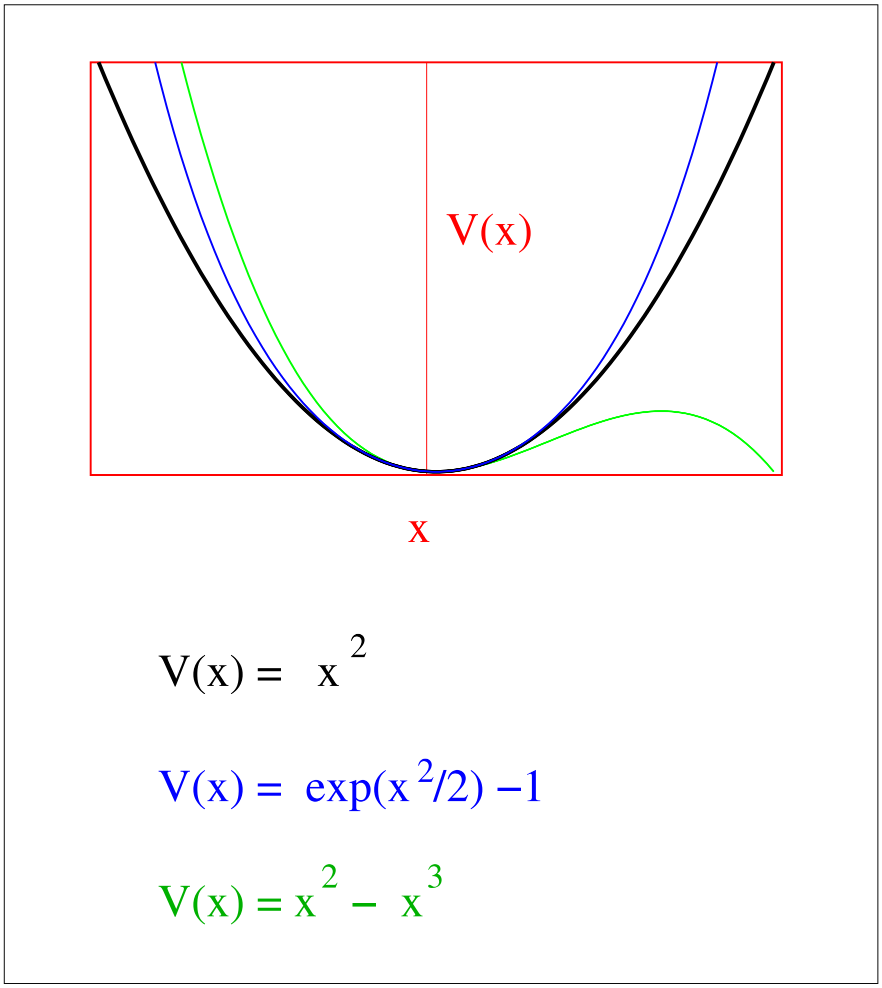

Figure 4: Various potentials approximated by harmonic oscillator near minima

where $\omega_0 = \sqrt{\frac{k}{m}}$ is known as *angular frequency.* Equation (4) represents a simple harmonic oscillator in its most general form.

$$\textit{Spring constant}: \quad k = m\omega_0^2 \tag{5}$$

$$\textit{Angular frequency}: \quad \omega_0 = \sqrt{k/m} \tag{6}$$

$$\textit{Time period}: \quad T = 2\pi/\omega_0 \tag{7}$$

$$\textit{Frequency}: \quad \nu = 1/T = \omega_0/2\pi \tag{8}$$

Examples:

1. Simple pendulum: The motion of simple pendulum is simple harmonic for small amplitudes (i.e. for small angular deviations)

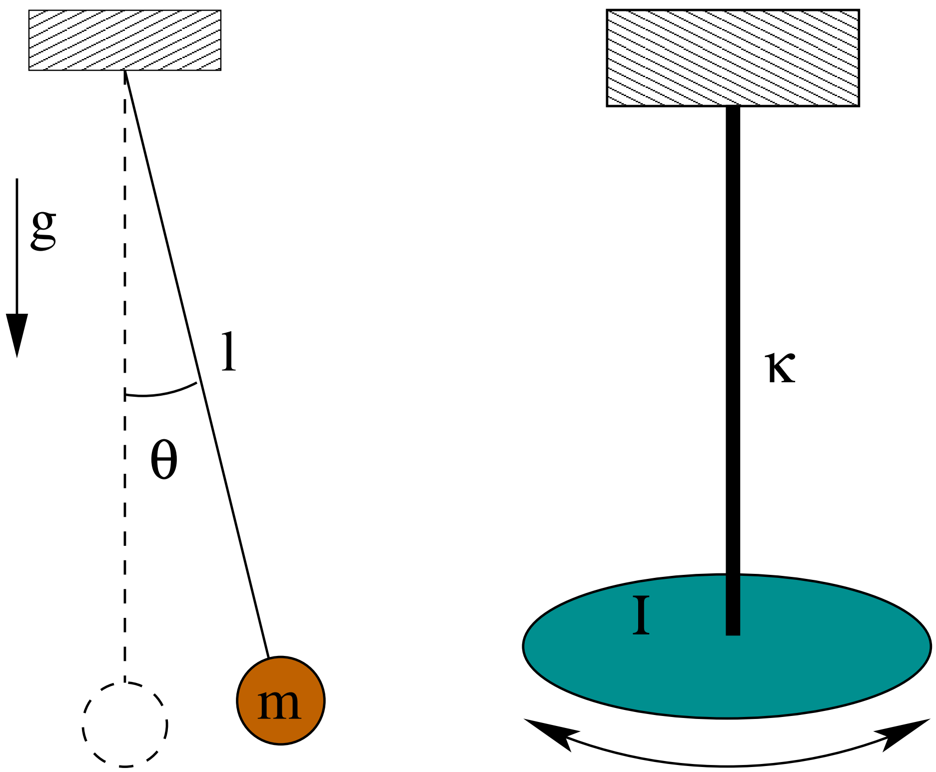

Figure 5: Simple and Torsional pendula

Equation of motion:

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta. \qquad (9)$$

For small θ,$\sin\theta \approx \theta$, and the equation (9) takes the following form,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0, \qquad (10)$$

giving $\omega_0 = \sqrt{\frac{g}{l}}$ and hence the time period $T = 2\pi\sqrt{\frac{l}{g}}$.

Note: If one takes $\theta \sim 15^\circ$ (so that the total angular amplitude is 30° and calculates the time period using the above SHM formula one makes an error of about 0.5%.

Problem: Show the above result.

2. Torsional pendulum: The equation for the torsional pendulum is the following.

$$I\frac{d^2\theta}{dt^2} + \kappa\theta = 0, \qquad (11)$$

where I is the moment of inertia of the object undergoing torsional oscillation about the axis of rotation and κ is the torsional constant. Angular frequency can be read

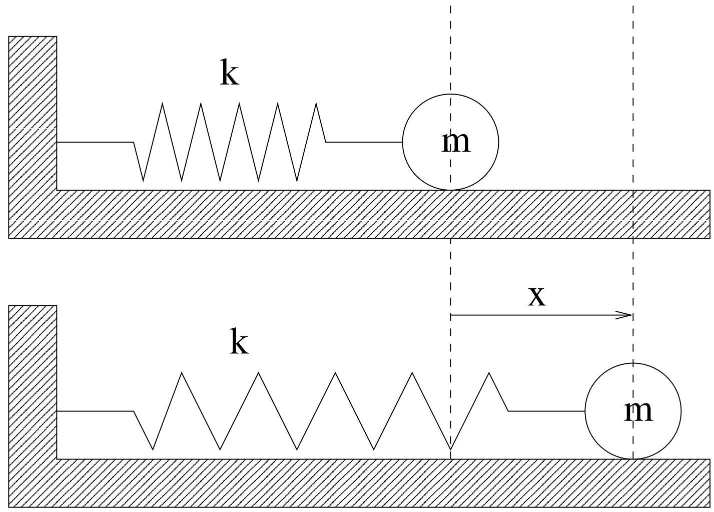

Figure 6: Spring-mass system

off directly as $\omega_0 = \sqrt{\frac{\kappa}{I}}$ and hence the time period, $T = 2\pi\sqrt{\frac{I}{\kappa}}$.

3. Spring-mass system: The equation of motion is same as the one shown in equation (4). The time period is given by $T = 2\pi\sqrt{\frac{m}{k}}$.

4. Physical pendulum or Compound pendulum: The equation of motion is,

$$I\frac{d^2\theta}{dt^2} = -Mgd\sin\theta, \tag{12}$$

where I is the moment of inertia about an axis perpendicular to the plane of oscillations through the point of suspension. For small oscillations($\theta < 4^\circ$) one can write the above equation (13) as

$$\frac{d^2\theta}{dt^2} + \frac{Mgd}{I}\theta = 0. \tag{13}$$

The above gives time period as $T = 2\pi\sqrt{\frac{I}{Mgd}}$.

Problem: Obtain simple pendulum results as a special case of the above.

5. L-C circuit:

If a capacitor is charged and then its two ends are connected with an inductance, the charge on the capacitor executes simple harmonic oscillations. The equation for the circuit is.

$$L\frac{di}{dt} = -q/C \tag{14}$$

where, i is the current in the circuit and q is the charge on the capacitor. L and C are inductance and capacitance respectively. Taking the current (i) as time derivative of (q) we obtain a simple harmonic oscillator equation.

$$L\frac{d^2q}{dt^2} + q/C = 0 \tag{15}$$

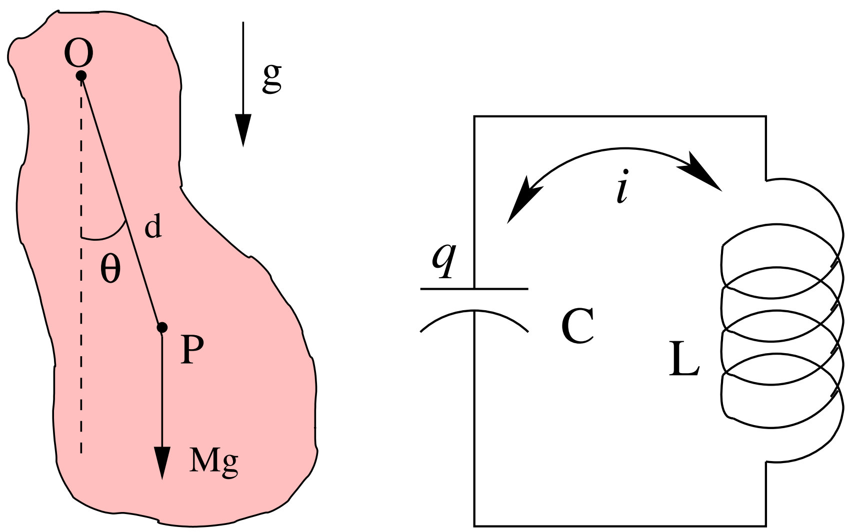

Figure 7: Compound pendulum and LC circuit

In this case $\omega_0 = 1/\sqrt{LC}$ and the time period is $T = 2\pi\sqrt{LC}$.

Solution: Solution of equation(4) is the following,

$$x = A\cos(\omega_0 t + \phi) \tag{16}$$

where A is the amplitude and ϕ is the phase respectively. These two quantities are decided by the two initial conditions, viz. the velocity, $\dot{x}(0)$ and position, $x(0)$, of the particle at some reference time $t = 0$. The solution (17) can be also written in another familiar form,

$$x = B\cos\omega_0 t + C\sin\omega_0 t. \tag{17}$$

In terms of phase ϕ, $B = A\cos\phi$ and $C = -A\sin\phi$.

One can easily see that $z(t) = A\exp(i\omega_0 t)$ with complex amplitude $A = |A|\exp(i\phi)$ also satisfies the equation of SHM. One can take the solution $x(t)$ as the real part of $z(t)$.

$$x(t) = \mathrm{Re}z(t) \tag{18}$$

The exponential solution is very easy to handle due to its nice properties of differentiation and multiplications. So addition of SHMs become very simple in this way. Suppose we have two SHMs $z_1(t) = A_1\exp(i\omega_0 t)$ and $z_2(t) = A_2\exp(i\omega_0 t)$ with $A_1 = |A|\exp(i\phi_1)$ and $A_2 = |A|\exp(i\phi_2)$ respectively. The resultant,

$$z(t) = z_1(t) + z_2(t) = |A|(\exp(i\phi_1) + \exp(i\phi_2))\exp(i\omega_0 t) \tag{19}$$

Following are the plots of displacement and velocity of the particle described by the equation, (17) for amplitude, $A = 2$ units, phase, $\phi = 30°$ and angular frequency, $\omega_0 = \pi$ rad/sec.

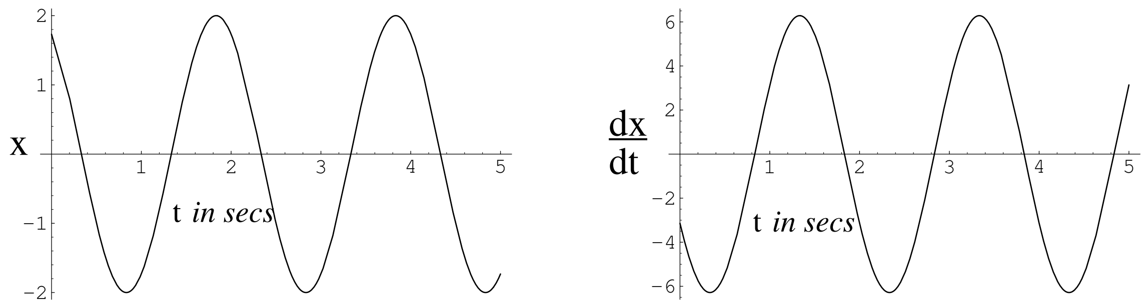

Figure 8: Displacement and velocity as a function of time

Energy of simple harmonic oscillator: The kinetic energy at any instant is given by,

$$k.e. \; = \; p^2/2m = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi) = \frac{1}{2}kA^2 \sin^2(\omega_0 t + \phi) \tag{20}$$

and the potential energy would be,

$$p.e. \; = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega_0 t + \phi) = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t + \phi) \tag{21}$$

The total energy, E of the system is a constant of motion and it is given by,

$$E = k.e. + p.e. = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2 \tag{22}$$

Phase space: The system is described completely by a position, x and a velocity, $\dot{x}$ (or momentum $p = m\dot{x}$) at each instant of time. The space of x and p is known as the "*phase space*." Evolution of a particle is described in this space by a "phase space diagram." The phase space diagram of a simple harmonic oscillator is an ellipse.

$$E = \frac{1}{2}kA^2 \;\; = \;\; p^2/2m + \frac{1}{2}kx^2 \tag{23}$$

$$\text{or, } 1 \;\; = \;\; \frac{p^2}{m^2\omega_0^2 A^2} + \frac{x^2}{A^2} \tag{24}$$

The last equation is an equation of ellipse with semi-axes A and $m\omega_0 A$.

Uniform circular motion and simple harmonic motion: Consider a particle moving on a circular path ($x^2 + y^2 = A^2$) with uniform angular velocity, ω. Now it is easy to see that the individual cartesian coordinates x and y of the particle perform SHM,

$$x = A\cos(\omega t) \qquad \text{and} \qquad y = A\sin(\omega t). \tag{25}$$

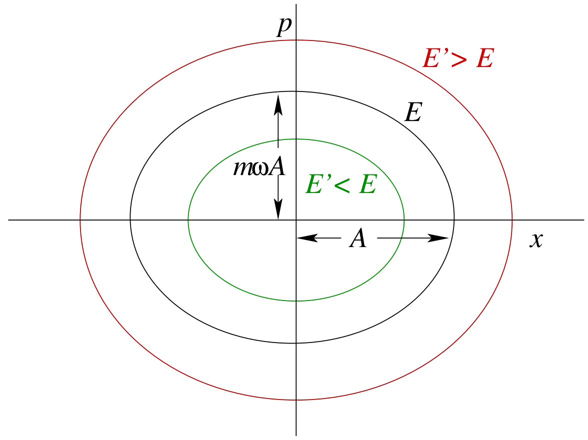

Figure 9: Phase space diagram of the SHO, with energies E and $E'(< E$ or $> E)$

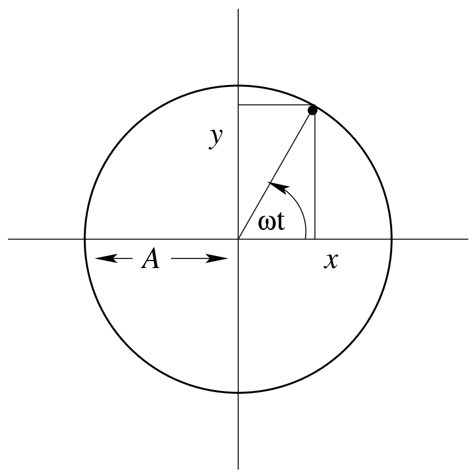

Figure 10: Uniform circular motion and SHM

If at $t = 0$ the particle's $y-$coordinate is non zero then the above equations are generalised with an initial phase ϕ,

$$x = A\cos(\omega t + \phi) \qquad \text{and} \qquad y = A\sin(\omega t + \phi). \tag{26}$$

Superposition of two simple harmonic oscillations in orthogonal direction:

A particle is under the influence of two SHMs in perpendicular directions. The displacements for the resulting motion can be written as,

$$x(t) \;\; = \;\; a\cos(\omega_1 t + \phi_1)$$

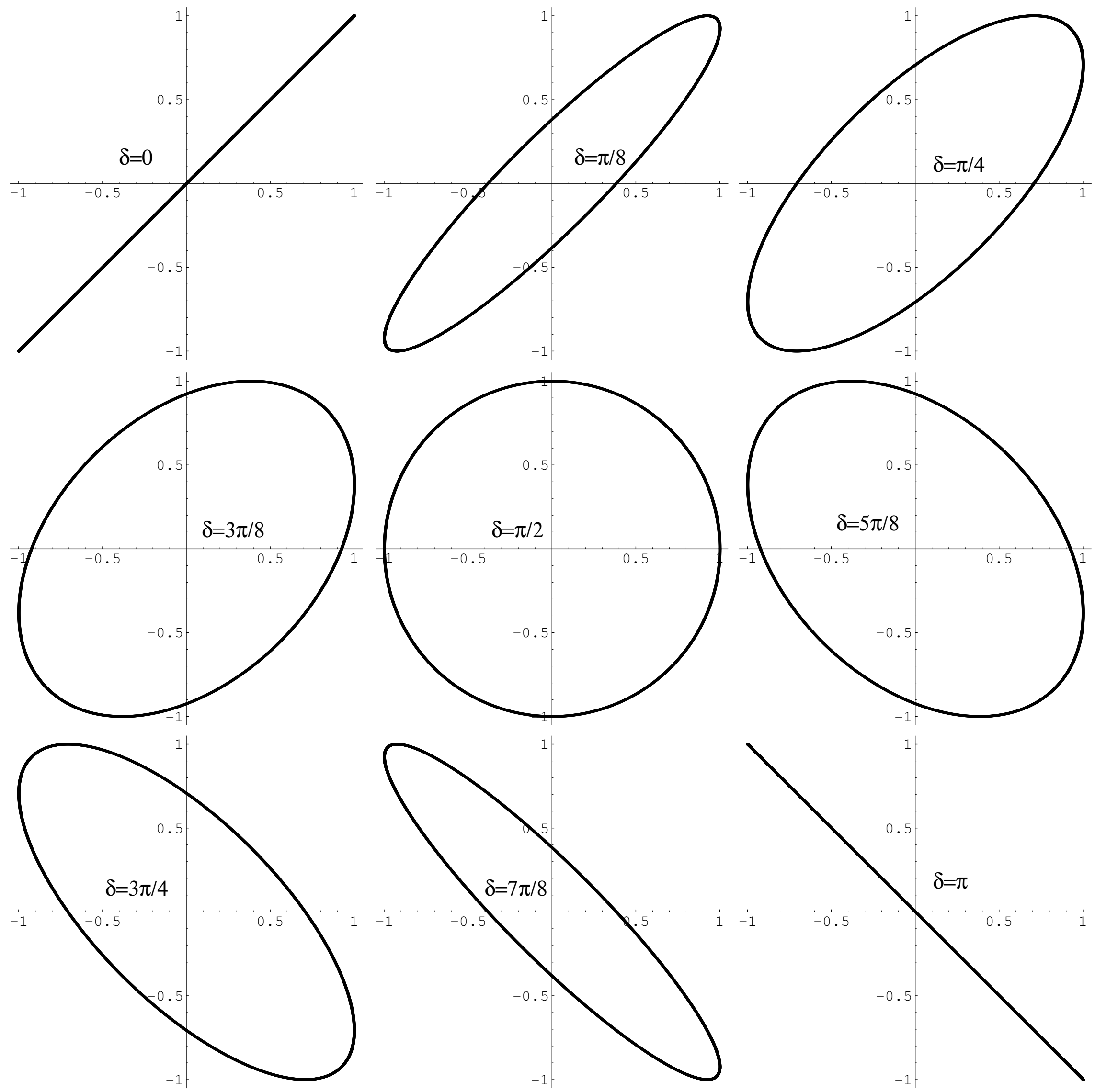

Figure 11: Superposition of two SHMs of equal frequencies and amplitudes in orthogonal directions

$$y(t) \quad = \quad b\cos(\omega_2 t + \phi_2) \tag{27}$$

The path of he particle could be obtained by eliminating the time t from the above equations. We first study the case when $\omega_1 = \omega_2$.
a) $\boldsymbol{\omega_1 = \omega_2} = \omega$: From equations (27) we have the following,

$$\left(\frac{x}{a}\cos\phi_2 - \frac{y}{b}\cos\phi_1\right) \quad = \quad \sin(\omega t)\sin(\phi_2 - \phi_1) \tag{28}$$

$$\left(\frac{x}{a}\sin\phi_2 - \frac{y}{b}\sin\phi_1\right) \quad = \quad \cos(\omega t)\sin(\phi_2 - \phi_1). \tag{29}$$

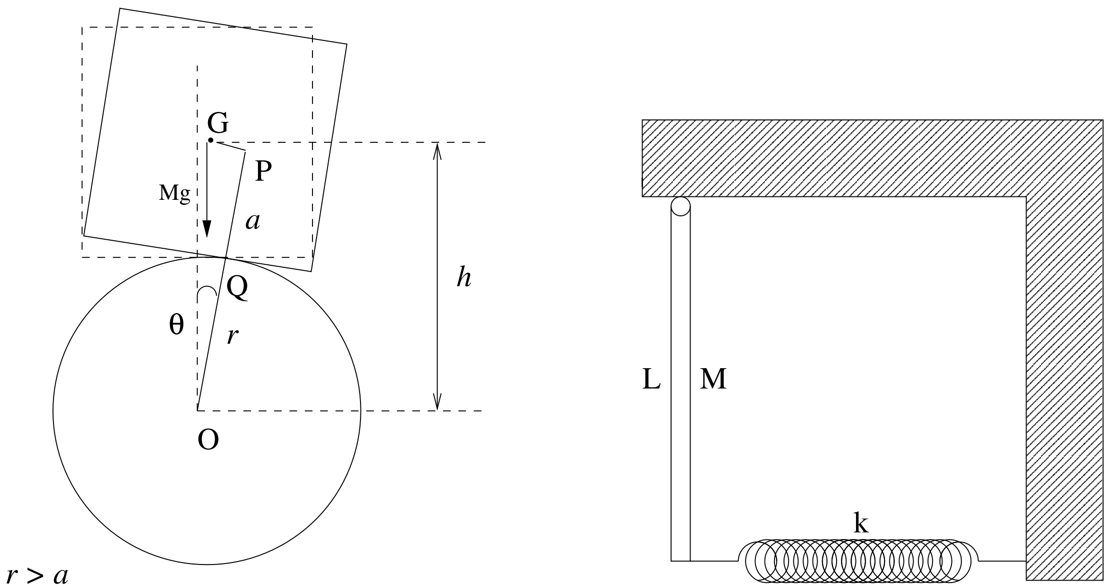

Figure 12: Diagrams for Example 1 and Problem 6.

Squaring and adding above two equations would eliminate t and we would get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1) \tag{30}$$

It is easy to see that the equation (30) represents a straight line for $\delta = \phi_2 - \phi_1 = \pm n\pi$, where n is an integer. For other values of δ it describes an ellipse. A circle could be obtained if one fixes $\delta = \pi/2$ and $a = b$. In figure (11) we show this superposition for $\omega_1 = \omega_2$, $a = b$ and various δ.

b) $\boldsymbol{\omega_1 \neq \omega_2}$: The cases where two frequencies are unequal the trajectories become complicated. These trajectories in $x - y$ are known as Lissajous figures. There are special cases of interest where one of the frequencies is an integer multiple of the other. In the following we give some of the Lissajous figures.

Example 1: A hard rubber cylinder of radius r is held fixed with its axis horizontal and a wooden cube of side $2a$ $(r > a)$ and mass M rests on it with its centre of mass vertically above the centre of cylinder as shown in the figure (12). The cube does not slip but it can rock on the cylinder. Find the time period of small oscillations if the cube is disturbed slightly from its stable position.

Solution 1: Since the cube does not slip but rocks we calculate torque about Q, the point where the cube is touching the cylinder. We can then write the equation of motion as,

$$I\ddot{\theta} = -Mg(r\theta\cos\theta - a\sin\theta) \tag{31}$$

where I is the moment of inertia of the cube about a horizontal axis passing through Q. For small θ we have the approximation $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ and hence the equation (31) reduces to the following simple harmonic oscillator equation.

$$\frac{5}{3}Ma^2\ddot{\theta} + Mg(r-a)\theta = 0 \tag{32}$$

Angular frequency can be directly read off from the equation (32) and time period of small oscillations is given by $T = 2\pi\sqrt{\frac{5a^2}{3(r-a)}}$. The above result shows that as r approaches a the time period becomes very large and oscillations cease.

Problem: Plot the potential energy of the cube as a function of θ for example 1. Taylor expand the potential near $\theta = 0$ and show that it is indeed harmonic oscillator potential. Are there equilibrium points other than $\theta = 0$? If yes, where are they located? What can you infer about their stability?

Problems

1. A simple pendulum of length l and mass m is suspended in a car that is travelling with a constant speed v around a circle of radius R. If the pendulum undergoes small oscillations about its equilibrium position, what will its frequency of oscillations be? [$\nu = \frac{1}{2\pi}\sqrt{\frac{(g^2+v^4/R^2)^{1/2}}{l}}$]

2. The scale of spring balance reading from 0 to 32 lb is 4.0 in long. A package suspended from the balance is found to oscillate vertically with a frequency of 2.0 oscillation per second. How much does the package weigh? [19 lb]

3. Calculate the period of small oscillations of a bottle which was slightly pushed down in the vertical direction in a liquid. The mass of the bottle is 150 gms and the radius of it is 2.5 cm, the density of the liquid is 1.0 gm/cm^3. The resistance of the liquid is assumed to be negligible. [T=0.55 sec]

4. Show that if a uniform stick of length l is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance d from the centre of mass, the period of oscillations has a minimum value when $d = l/\sqrt{12}$.

5. The potential energy of a one-dimensional mass at a distance r from the origin is $V(r) = V_0(\frac{r}{R} + \lambda^2\frac{R}{r})$, for $0 < r < \infty$ with V_0, R and λ all positive constants. What is the angular frequency of small oscillations around the stable position?[$\omega_0 = \sqrt{\frac{2V_0}{m\lambda R^2}}$]

6. A uniform thin rod of mass M and length L hangs from a frictionless pivot and is connected at the bottom by a spring to the wall as shown. The spring constant is k. What is the period of small oscillations? [$T = 2\pi\sqrt{\frac{2ML}{3(2k+Mg)}}$]

7. A solid cylinder of mass M is attached to a horizontal massless spring so that it can roll without slipping along a horizontal surface, as shown in figure. The force constant of the spring is k. If the system is released from the rest at a position in which the spring is stretched by a little amount show that the centre of mass of the cylinder executes simple harmonic oscillations with a period $T = 2\pi\sqrt{\frac{3M}{2k}}$.

8. Determine the period of small oscillations of mercury of mass 200gm poured into a bent tube as shown in the figure whose right arm forms an angle 30° with the

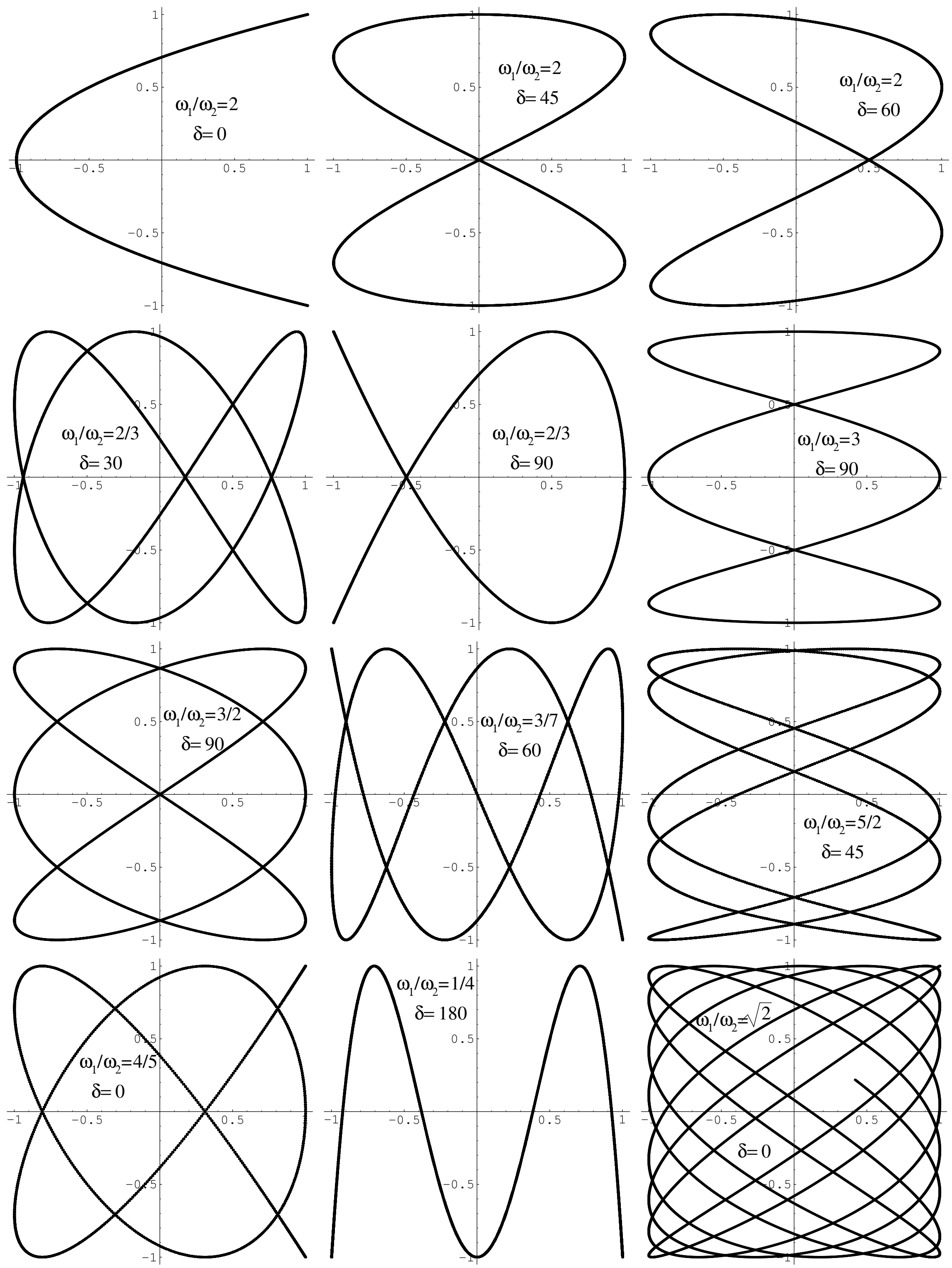

Figure 13: Lissajous figures

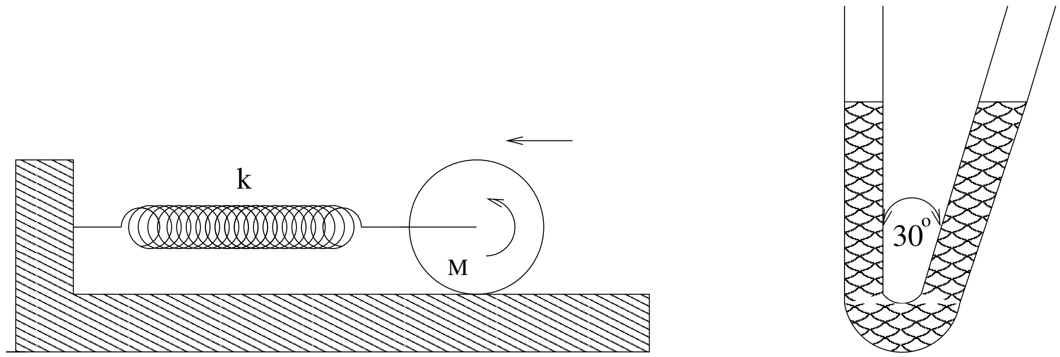

Figure 14: Diagrams for Problem 7 and Problem 8.

vertical. The cross-sectional area of the tube is 0.5 cm^2. The viscosity of mercury is to be neglected. [T=0.8 sec]

9. An 8.0 lb block is suspended from a spring with a force constant of 3.0 lb/in. A bullet weighing 0.10 lb is fired into the block from below with a velocity of 500 ft/sec and comes to rest in the block. Find the amplitude of the resulting simple harmonic motion. Use $g = 32$ ft/sec^2. [6.2 in]

10. A particle oscillates with simple harmonic motion along the x-axis with a displacement amplitude a and spends a time dt in moving x to $x + dx$.Show that the probability of finding it between x and $x + dx$ is given by $\frac{dx}{\pi(a^2-x^2)^{1/2}}$.

11. Consider oscillations described through

$$z(t) = A\exp(i\omega t),$$

where A is the complex amplitude. What are the initial position and velocity for a) $A = |A|$, b) $A = |A|\exp(i\pi/3)$, c) $A = |A|\exp(i\pi/2)$ and d) $A = |A|\exp(i2\pi/3)$.

12. Consider two oscillations of the same mass described through $z_1(t) = A_1\exp(i2\omega t)$ and $z_2(t) = A_2\exp(i3\omega t)$ with $A_1 = |A|\exp(i\pi/3)$ and $A_1 = |A|\exp(i2\pi/3)$. For both oscillators,

a) Plot the position as function of time and graphically determine how many times the two oscillations intersect before the first oscillator reaches the origin.

b) Are the masses moving in the same or opposite directions for each intersection?

c) Which oscillator has a larger kinetic energy at $t = 0$?

d) For which oscillator is the mean kinetic energy larger?