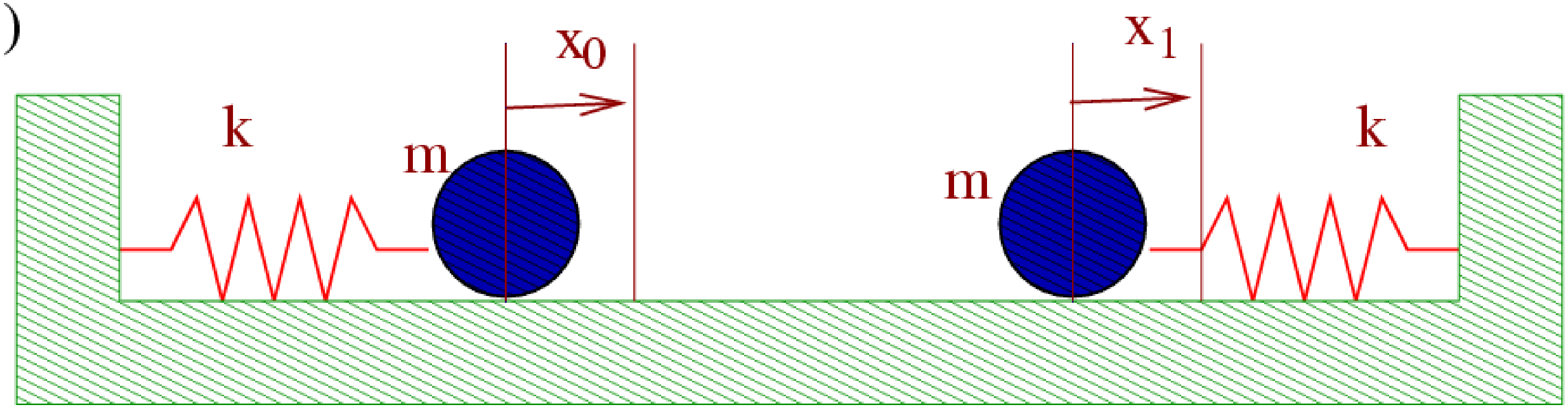
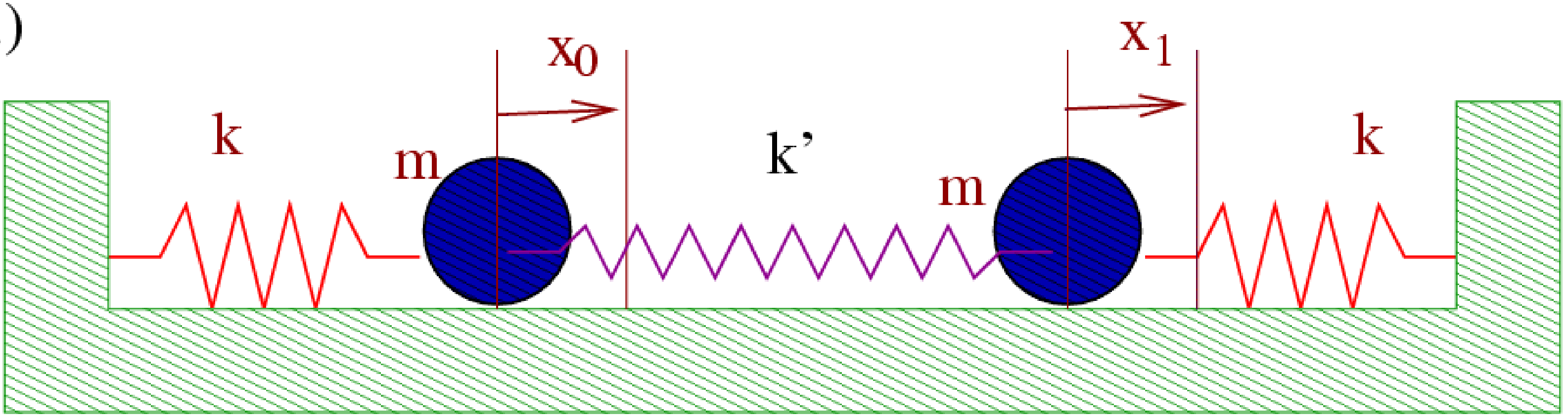
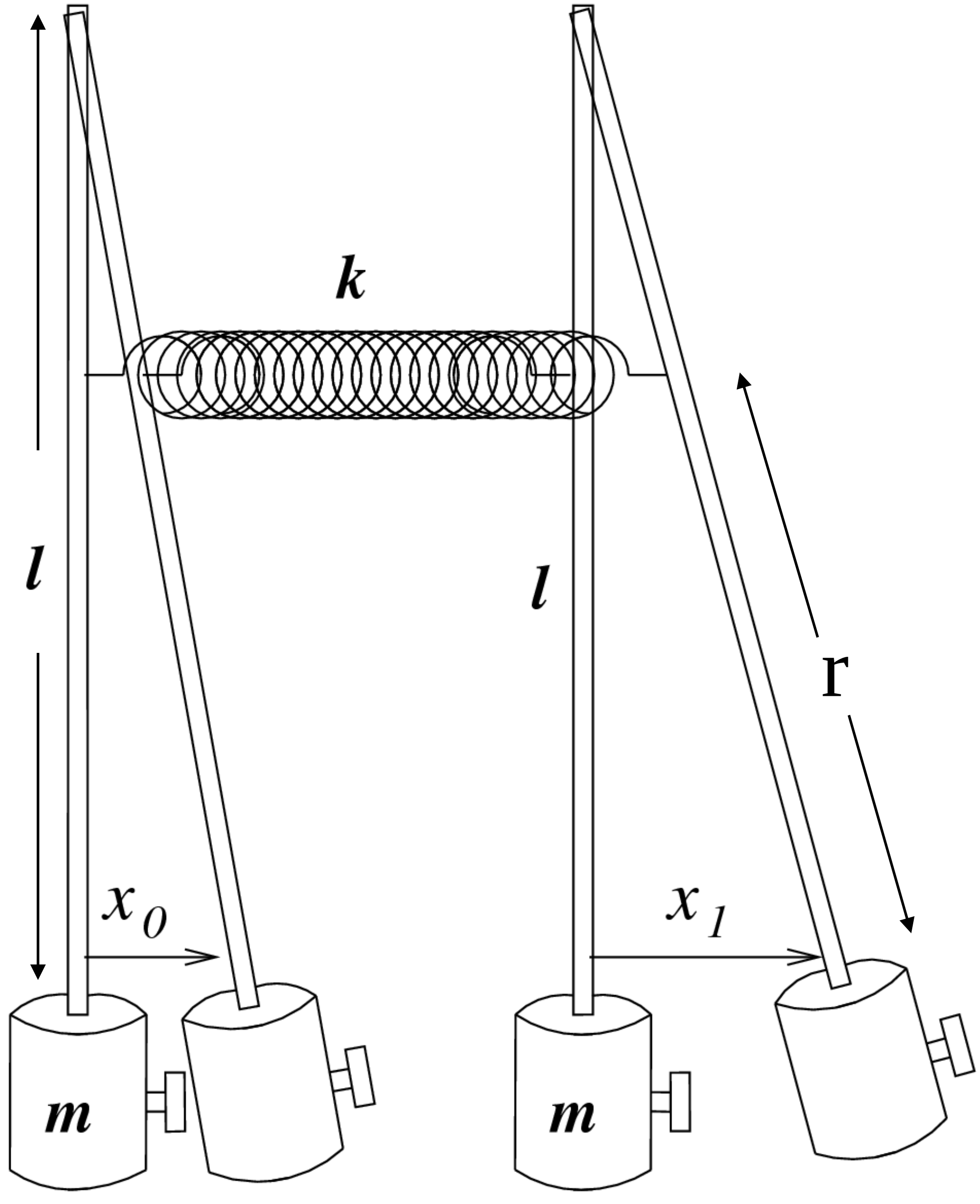
Coupled oscillators

(a.)



(b.)





Coupled pendula

Equations of motion

$$m\frac{d^2 x_0}{dt^2} = -kx_0 - k'(x_0 - x_1)$$

$$m\frac{d^2 x_1}{dt^2} = -kx_1 - k'(x_1 - x_0)$$

$$q_0 = \frac{x_0 + x_1}{2} \text{ and } q_1 = \frac{x_0 - x_1}{2}$$

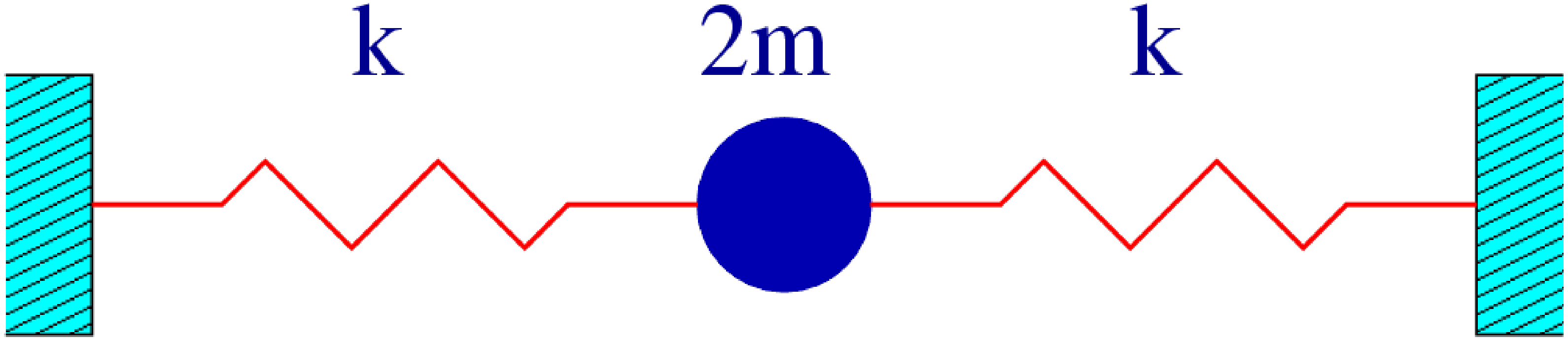
Normal modes

$$m\frac{d^2 q_0}{dt^2} = -kq_0$$

$$m\frac{d^2 q_1}{dt^2} = -(k + 2k^{'})q_1$$

Centre of mass

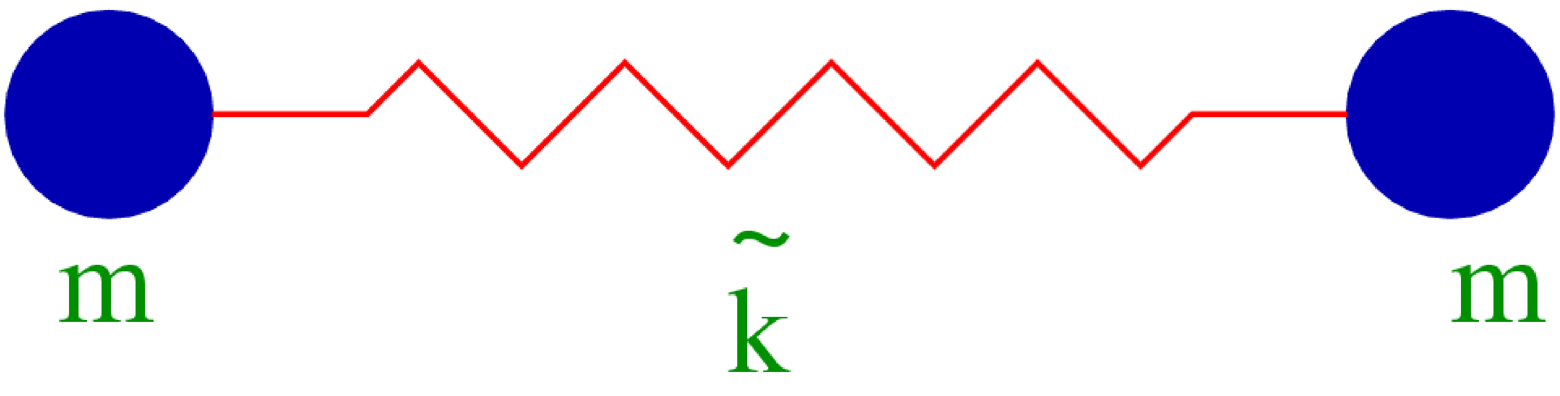
$$\omega_0 = \sqrt{\frac{2k}{2m}}$$

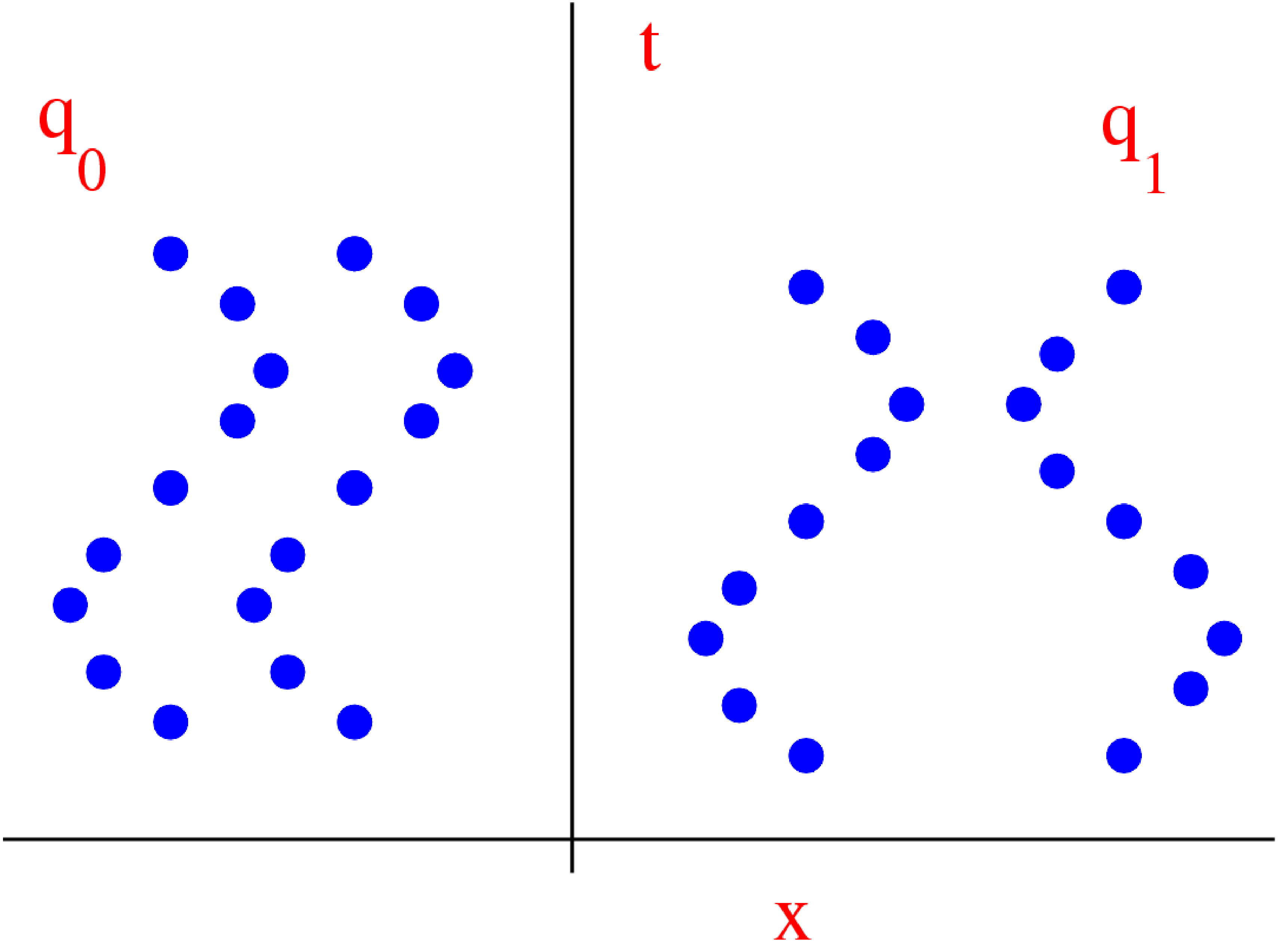


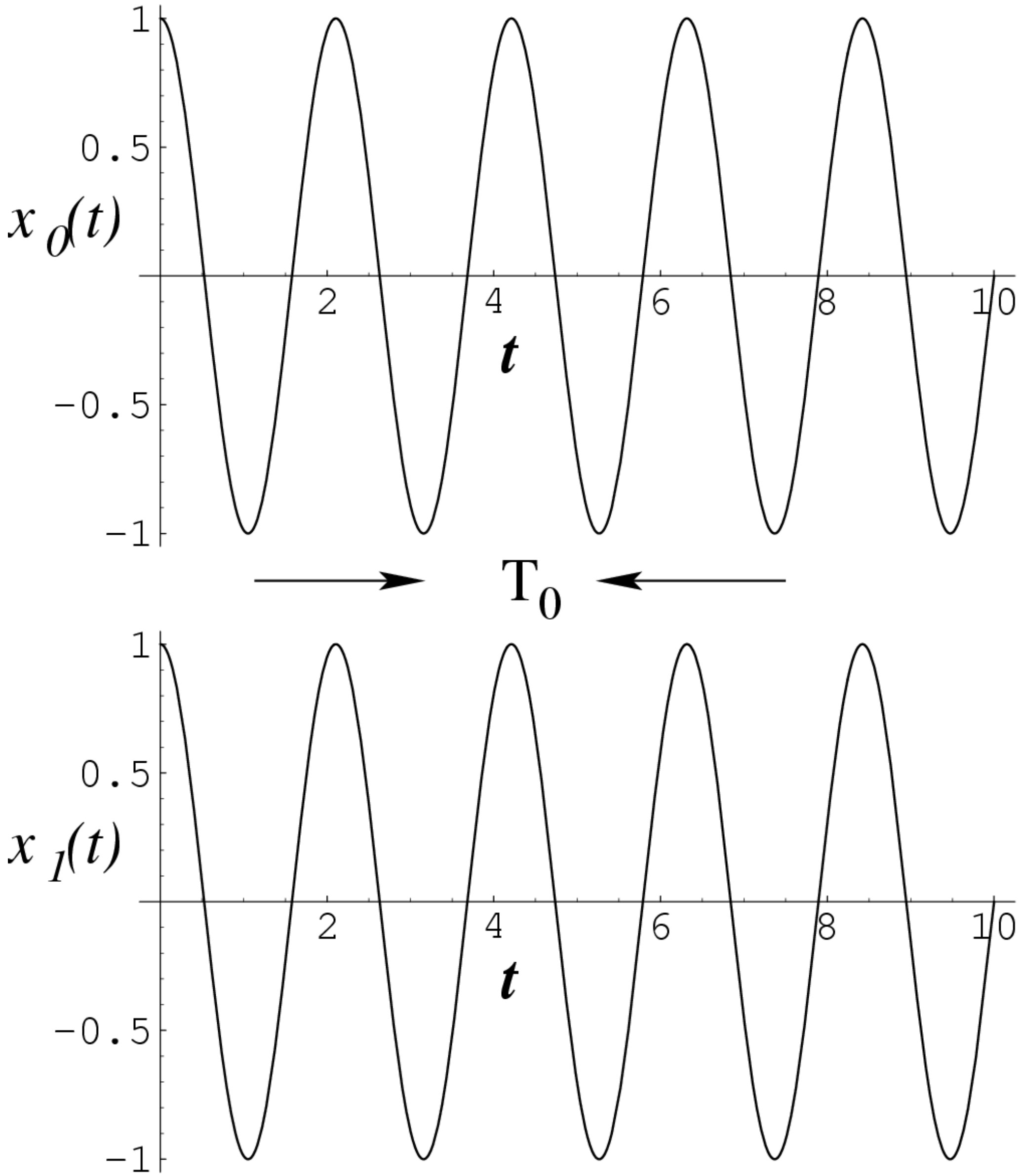
Relative coordinate

$$\tilde{k} = (k + 2k')/2$$

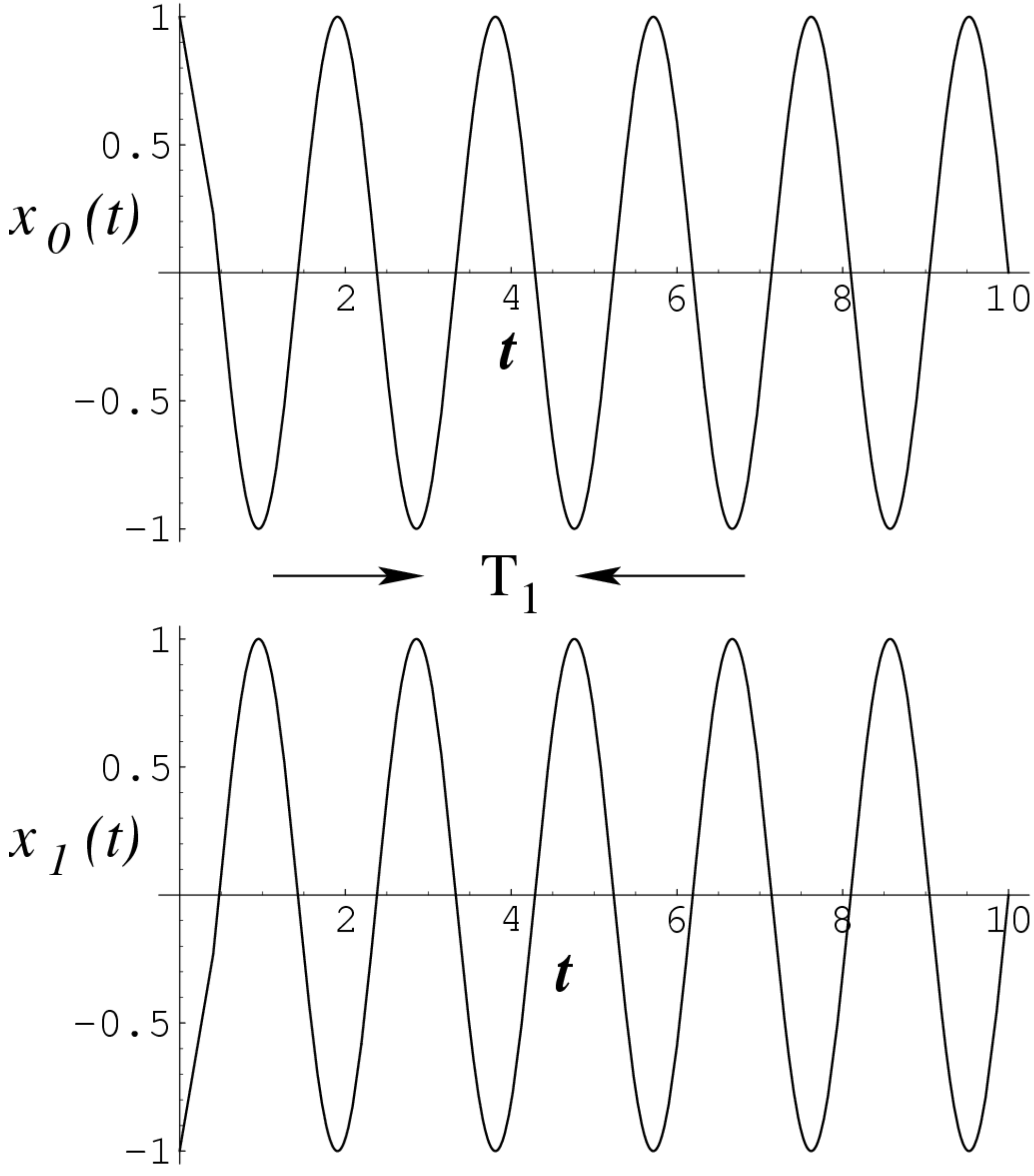
$$\omega_1 = \sqrt{\frac{k+2k'}{m}}$$




t
q_0
q_1
x



In−phase mode



Out−of−phase mode

$$q_0(t) = A_0 \ e^{i \ \omega_o t}$$

$$q_1(t) = A_1 \ e^{i \ \omega_1 t}$$

$$A_0 = |A_0| \, e^{i\psi_0} \quad \text{etc.}$$

$$x_0(t) = |A_0| \, e^{i(\omega_0 t + \psi_0)} + |A_1| \, e^{i(\omega_1 t + \psi_1)}$$

$$x_0(t) = A_0\ e^{i\ \omega_0 t} + A_1\ e^{i\ \omega_1 t}$$

$$x_1(t) = A_0\ e^{i\ \omega_0 t} - A_1 e^{i\ \omega_1 t}$$

$$x_1\ (0) = 0 \Rightarrow A_0 = A_1$$

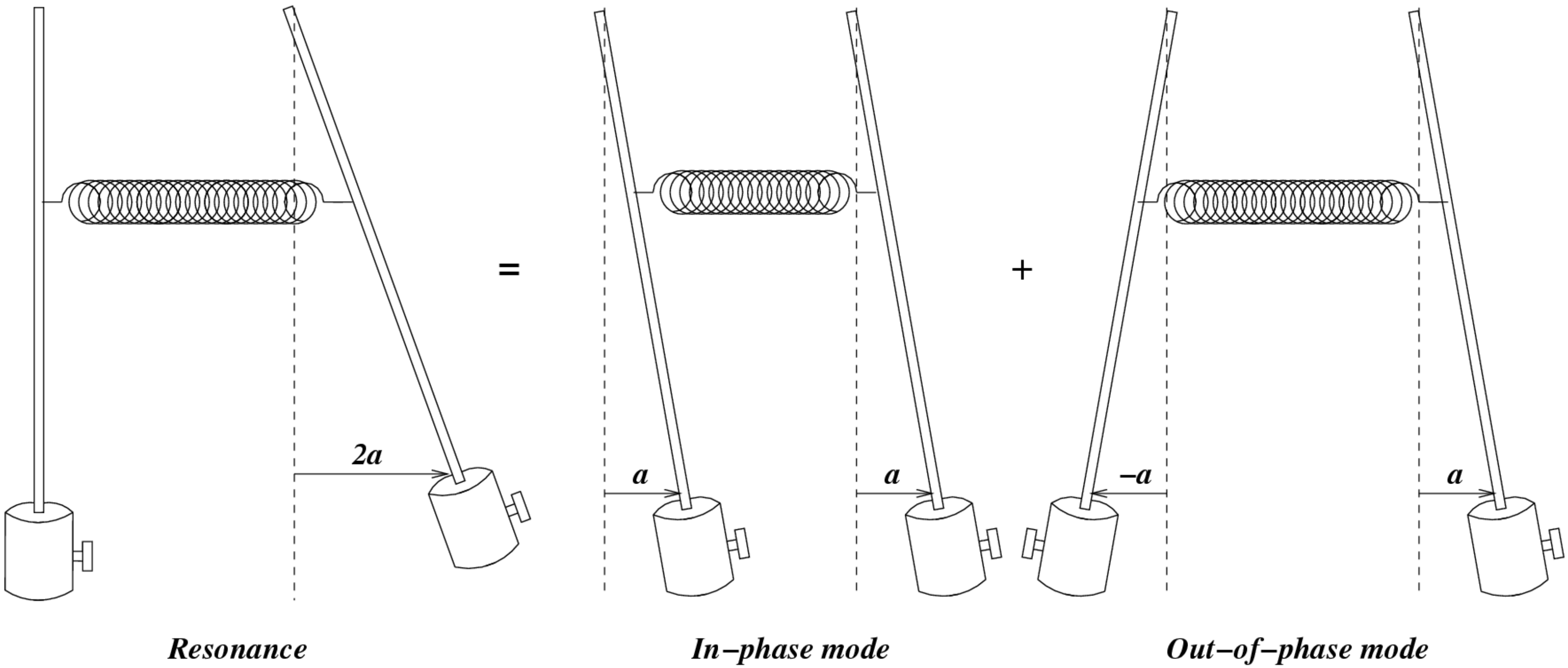
$$x_0(t) = A_0 \left[e^{i\ \omega_0 t} + e^{i\ \omega_1 t} \right]$$

$$\dot{x}_0(0) = 0 \Rightarrow \ \mathrm{Re}\left\{A_0\left[i\ \omega_0 + i\ \omega_1\right]\right\} = 0$$

$$\Rightarrow \ A_0 \ \text{is real}, \quad = \frac{a_0}{2}$$

$$x_0(t) = \frac{a_0}{2}\left[\cos\ \omega_0 t + \cos\ \omega_1 t\right]$$

$$x_1(t) = \frac{a_0}{2}\left[\cos\ \omega_0 t - \cos\ \omega_1 t\right]$$

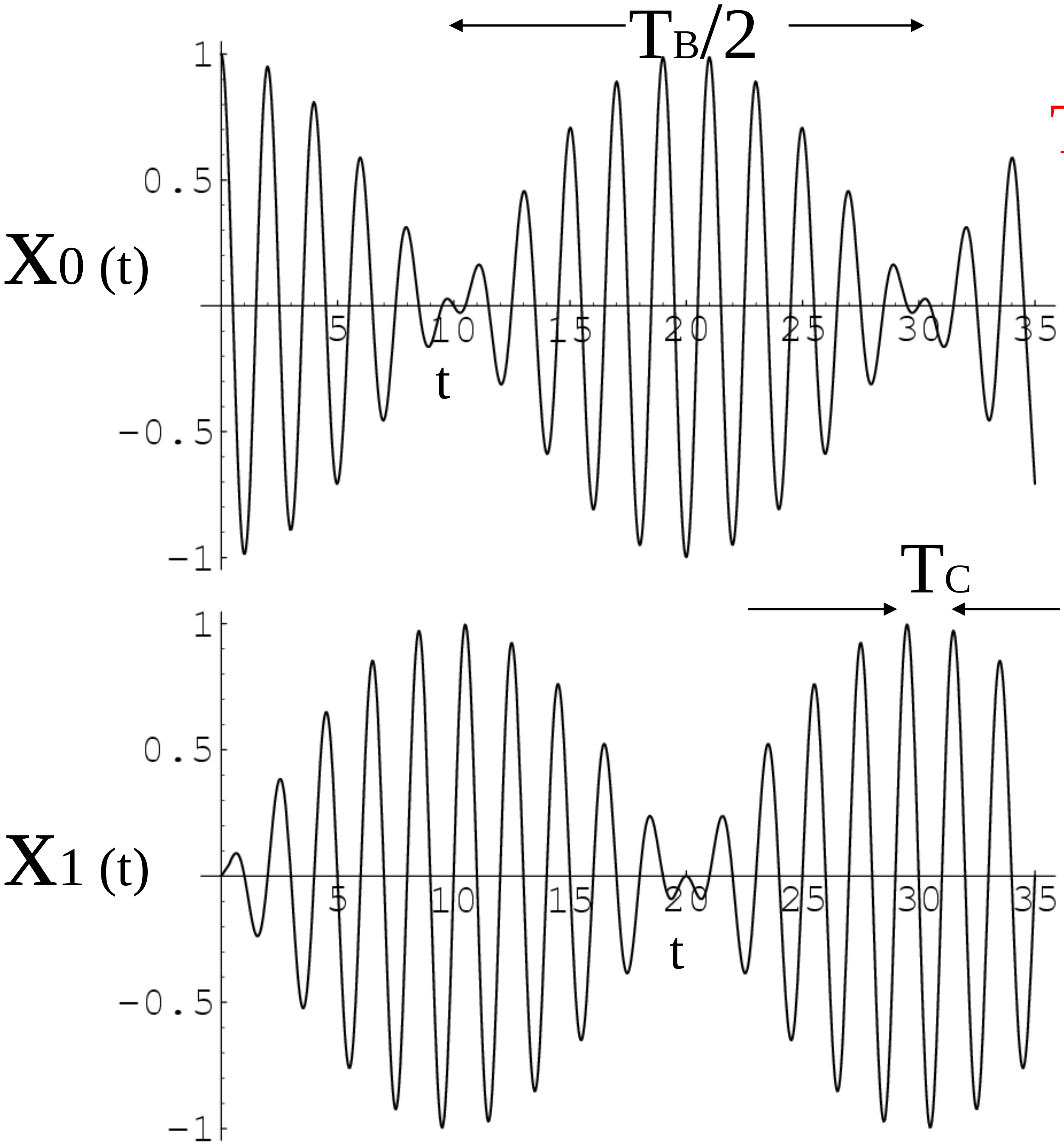

2a
=
a
a
+
−a
a
Resonance
In−phase mode
Out−of−phase mode

$$x_0(t) = a_0 \ \cos\left[\left(\frac{\omega_1 - \omega_0}{2}\right) t\right] \ \cdot$$

$$\cos\left[\left(\frac{\omega_0 + \omega_1}{2}\right) t\right]$$

$$x_1(t) = a_0 \ \sin\left[\frac{(\omega_1 - \omega_0)t}{2}\right] \ \cdot$$

$$\sin\left[\left(\frac{\omega_0 + \omega_1}{2}\right) t\right]$$



T_B=Beats Period

Coupled oscillations (Resonance)

Weak coupling $K' \ll K$

$$\omega_1 = \sqrt{\frac{K}{m}\left(1 + \frac{2K'}{K}\right)} \approx \omega_0 + 2\Delta\omega$$

$$\frac{\Delta\omega}{\omega_0} = \frac{K'}{2K} \ll 1$$

$$x_1(t) = a_0 \ \cos \ \Delta\omega t \ \cos \ \omega_0 t$$

$$x_2(t) = a_0 \ \sin \ \Delta\omega t \ \sin \ \omega_0 t$$

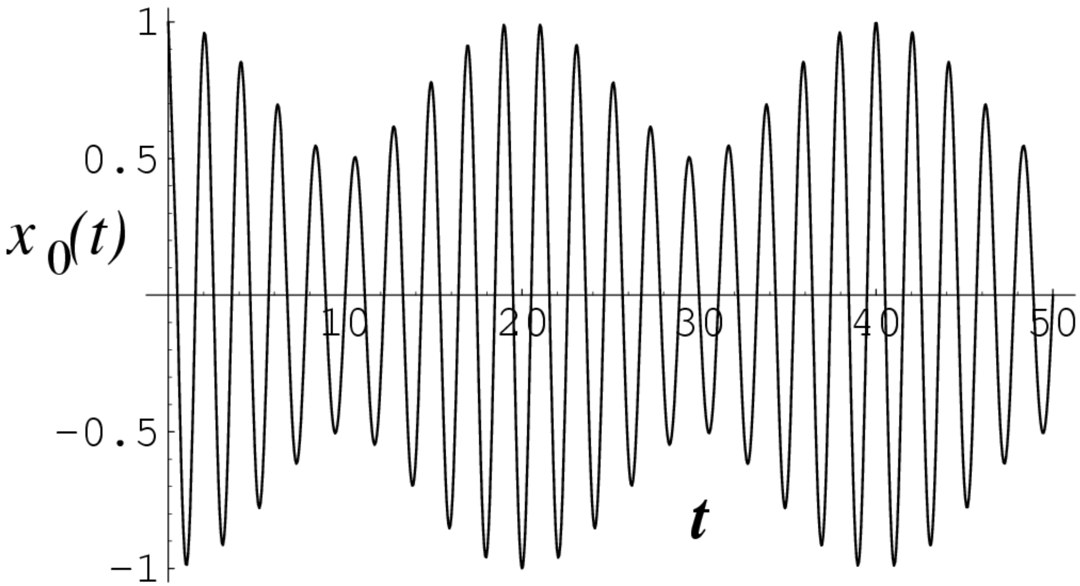
Stiff coupling $\quad K' \gg K$

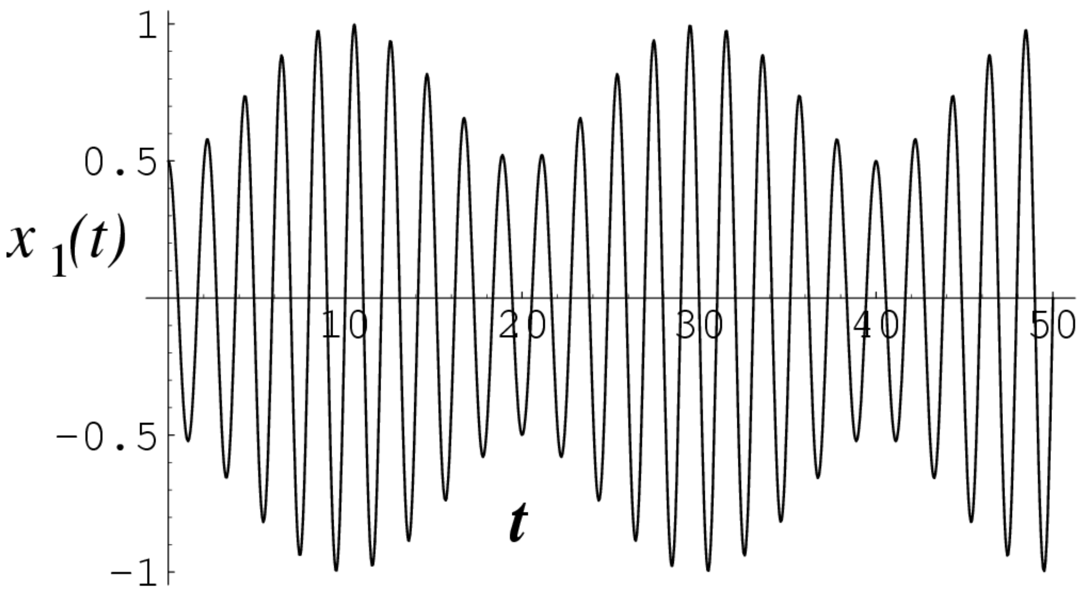
Connecting two masses with rigid rod

$$\omega_1 \gg \omega_0$$

$$\text{on a small time scale} \sim \frac{1}{\omega_1},$$

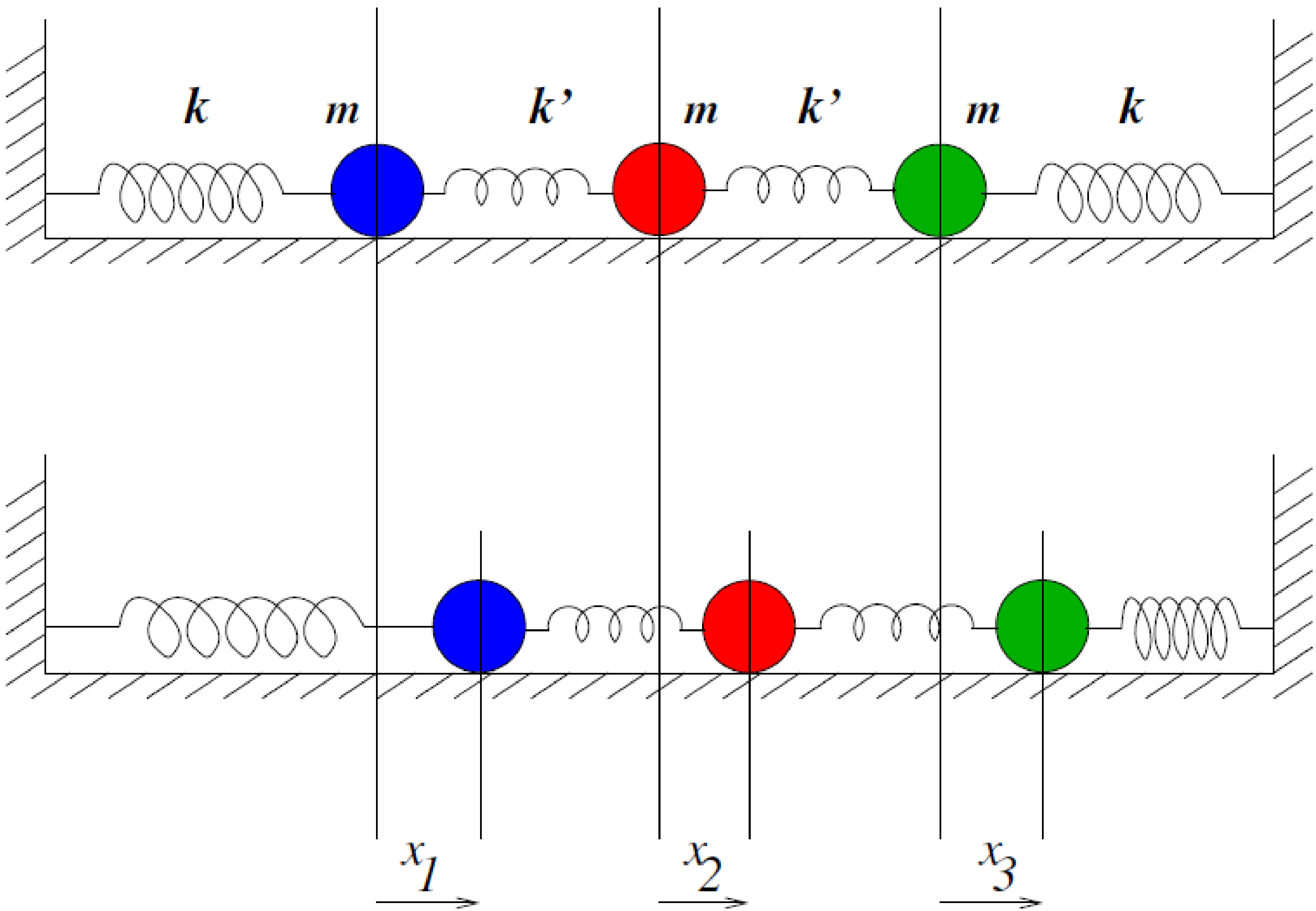
$$\text{we will not see the slow oscillation.}$$





Coupled Oscillations

Find the eigen frequencies of the normal modes for the following coupled oscillaors.



The equations of motion

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k'(x_1 - x_2) \\ m\ddot{x}_2 &= -k'(x_2 - x_1) - k'(x_2 - x_3) \\ m\ddot{x}_3 &= -k'(x_3 - x_2) - kx_3 \end{aligned}$$

For eigen/normal modes substitute

$$x_1 = A\exp(i\omega t); \qquad x_2 = B\exp(i\omega t); \qquad x_3 = C\exp(i\omega t).$$

and write the equations in a matrix form, like the following

$$\begin{bmatrix} m\omega^2 - k - k' & k' & 0 \\ k' & m\omega^2 - 2k' & k' \\ 0 & k' & m\omega^2 - k - k' \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = 0$$

Condition for the existence of solutions

$$\begin{vmatrix} m\omega^2 - k - k' & k' & 0 \\ k' & m\omega^2 - 2k' & k' \\ 0 & k' & m\omega^2 - k - k' \end{vmatrix} = 0$$

Giving squares of eigen frequencies as

$$\begin{aligned} \omega_0^2 &= \frac{1}{m}(k + k'), \\ \omega_\pm^2 &= \frac{1}{2m}\left[(k + 3k') \pm \sqrt{(k + 3k')^2 - 8kk'}\right]. \end{aligned}$$