Forced oscillator:Damped case

$$m\ddot{x} + 2r\dot{x} + kx = F(t)$$
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$$

$F(t) =$ Time dependent function

$$F(t) = F_0 \cos \omega t$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t = f_0 \cos \omega t$$

Companion equation:

$$F(t) = F_0 \sin \omega t,$$

$$\ddot{y} + 2\beta \dot{y} + \omega_0^2 y = \frac{F_0}{m} \sin \omega t = f_0 \sin \omega t$$

$$z = x + iy$$

$$\ddot{z} + 2\beta \dot{z} + \omega_0^2 z = \frac{F_0}{m} \exp(i\omega t) = f_0 \exp(i\omega t)$$

$$\ddot{x}_1 + 2\beta\dot{x}_1 + \omega_0^2 x_1 = F_1(t)/m$$
$$\ddot{x}_2 + 2\beta\dot{x}_2 + \omega_0^2 x_2 = F_2(t)/m$$
$$\ddot{x}_3 + 2\beta\dot{x}_3 + \omega_0^2 x_3 = F_3(t)/m$$
$$\cdots\cdots$$
$$\sum_i \ddot{x}_i + 2\beta \sum_i \dot{x}_i + \omega_0^2 \sum_i x_i = \sum_i F_i(t)/m$$
$$\boxed{\ddot{u} + 2\beta\dot{u} + \omega_0^2 u = F(t)/m}$$

$u = \sum_i x_i$ and $F(t) = \sum_i F_i(t)$

Complementary function: Transients

Try steady state solution (*Particular solution*)

$$z(t) = z_0 \exp(i\omega t)$$

Find z_0

Complementary function: Transients

Try steady state solution (*Particular solution*)

$$z(t) = z_0 \exp(i\omega t)$$

Find z_0

$$z_0(-\omega^2 + 2i\beta\omega + \omega_0^2) = f_0$$

$$z_0 = \frac{f_0}{(\omega_0^2 - \omega^2 + 2i\beta\omega)}$$

$$z_0 = |z_0| \exp(i\phi)$$

$$|z_0| = \sqrt{z_0 z_0^*} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\phi = \tan^{-1}\left(\frac{-2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

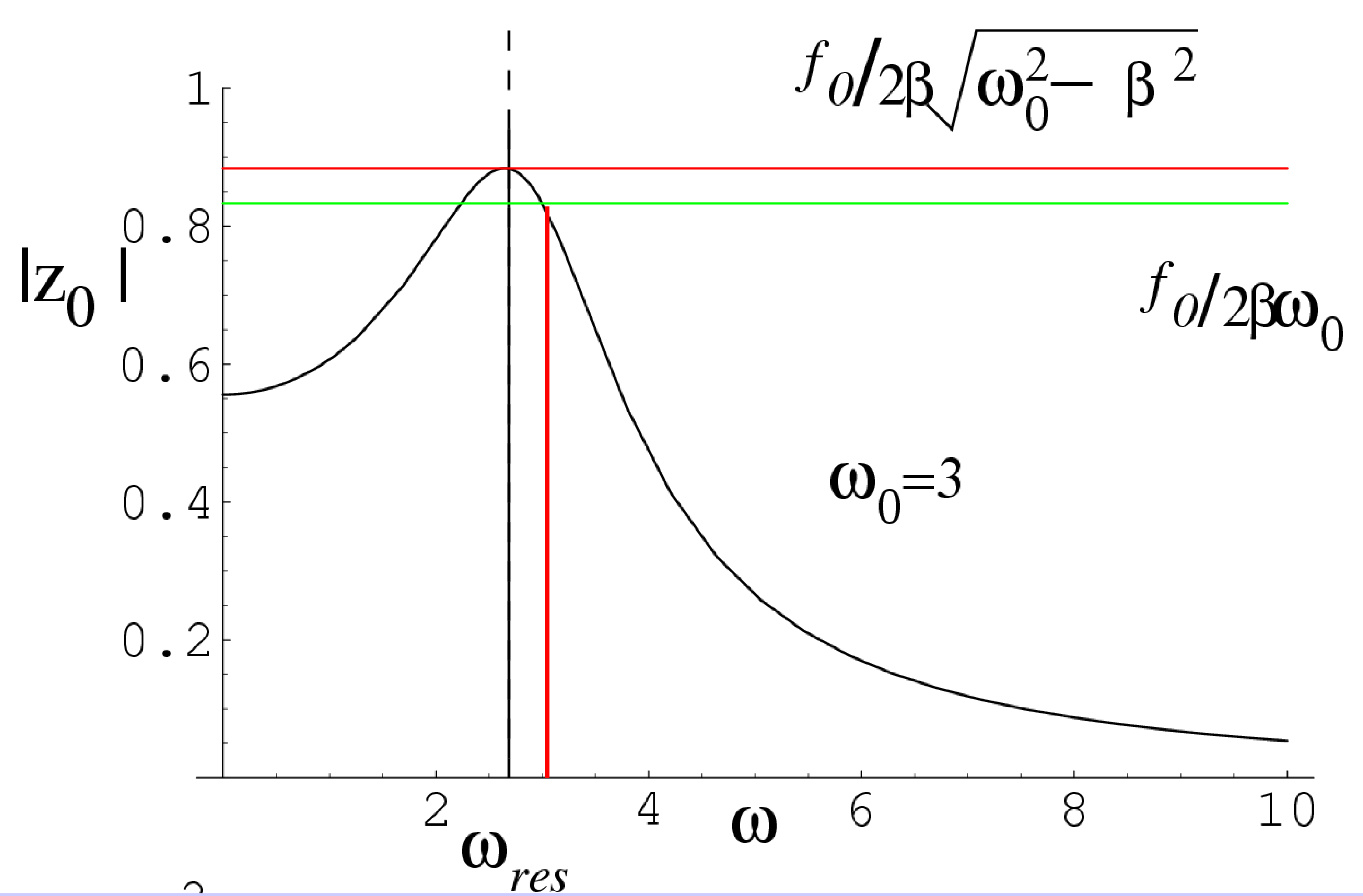
$$z(t) = |z_0| \exp(i(\omega t + \phi))$$

$$= |z_0|(\cos(\omega t + \phi) + i \sin(\omega t + \phi))$$

$$x(t) = |z_0| \cos(\omega t + \phi)$$

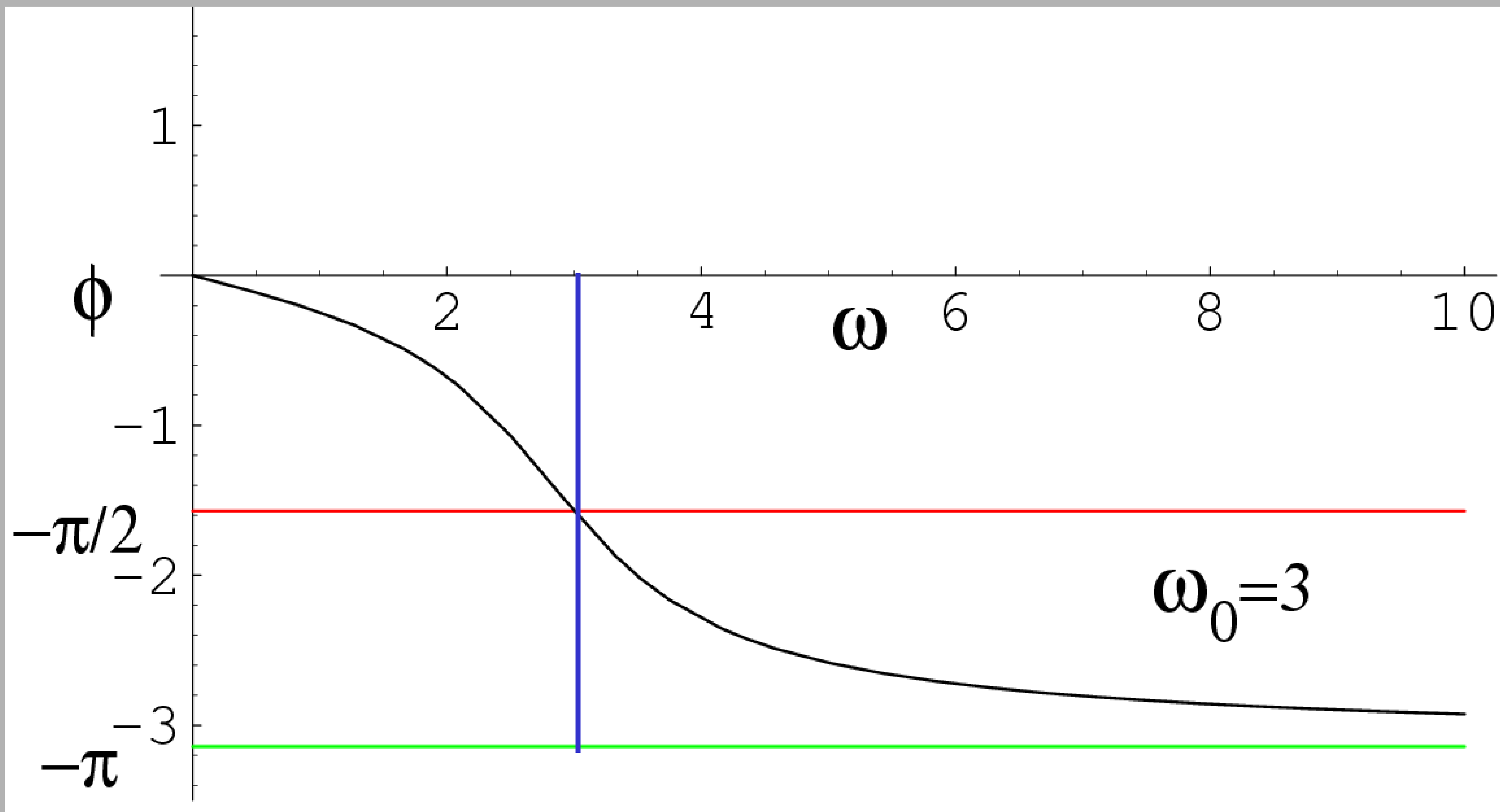
$$y(t) = |z_0| \sin(\omega t + \phi)$$

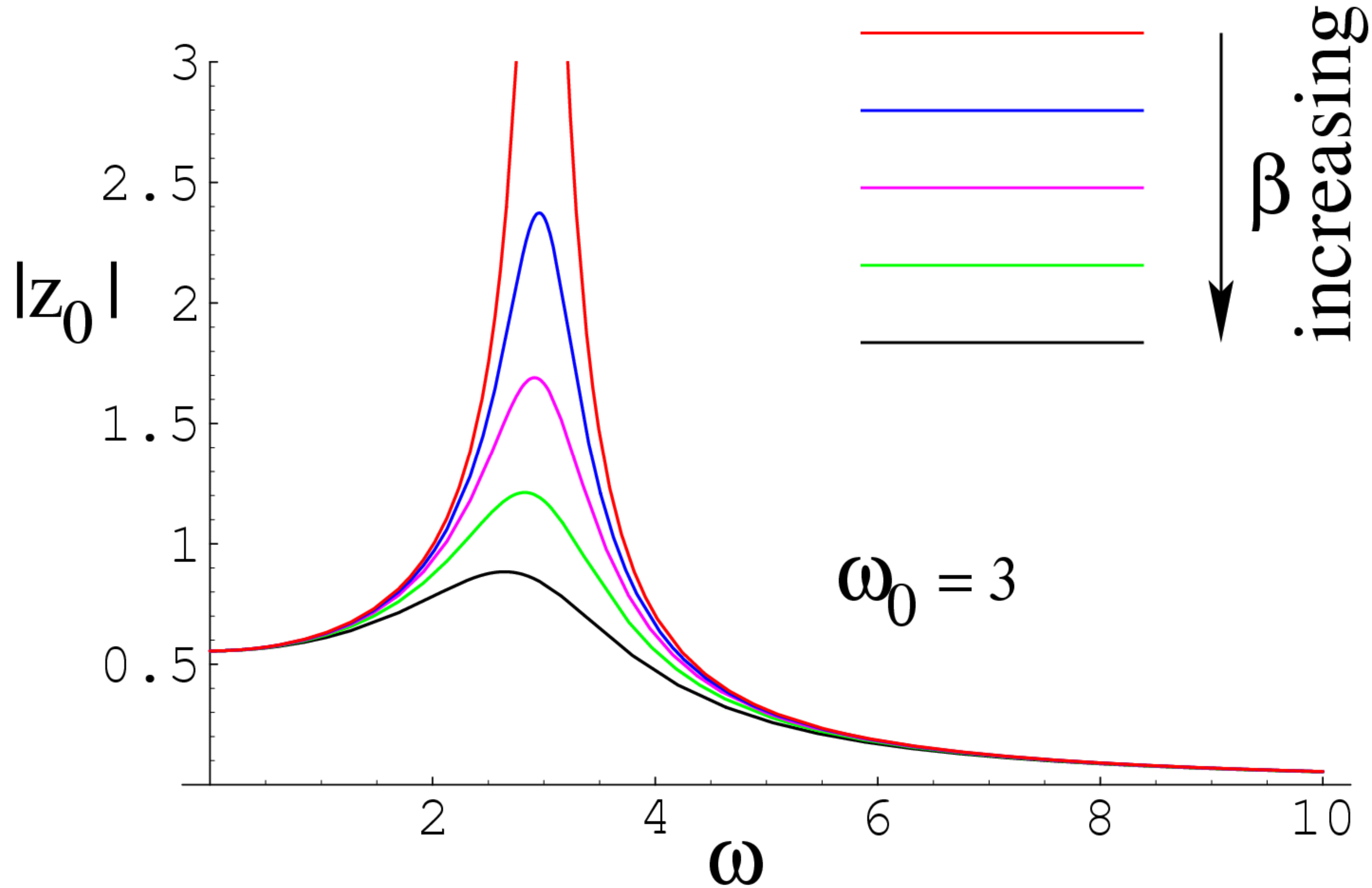
Amplitude:



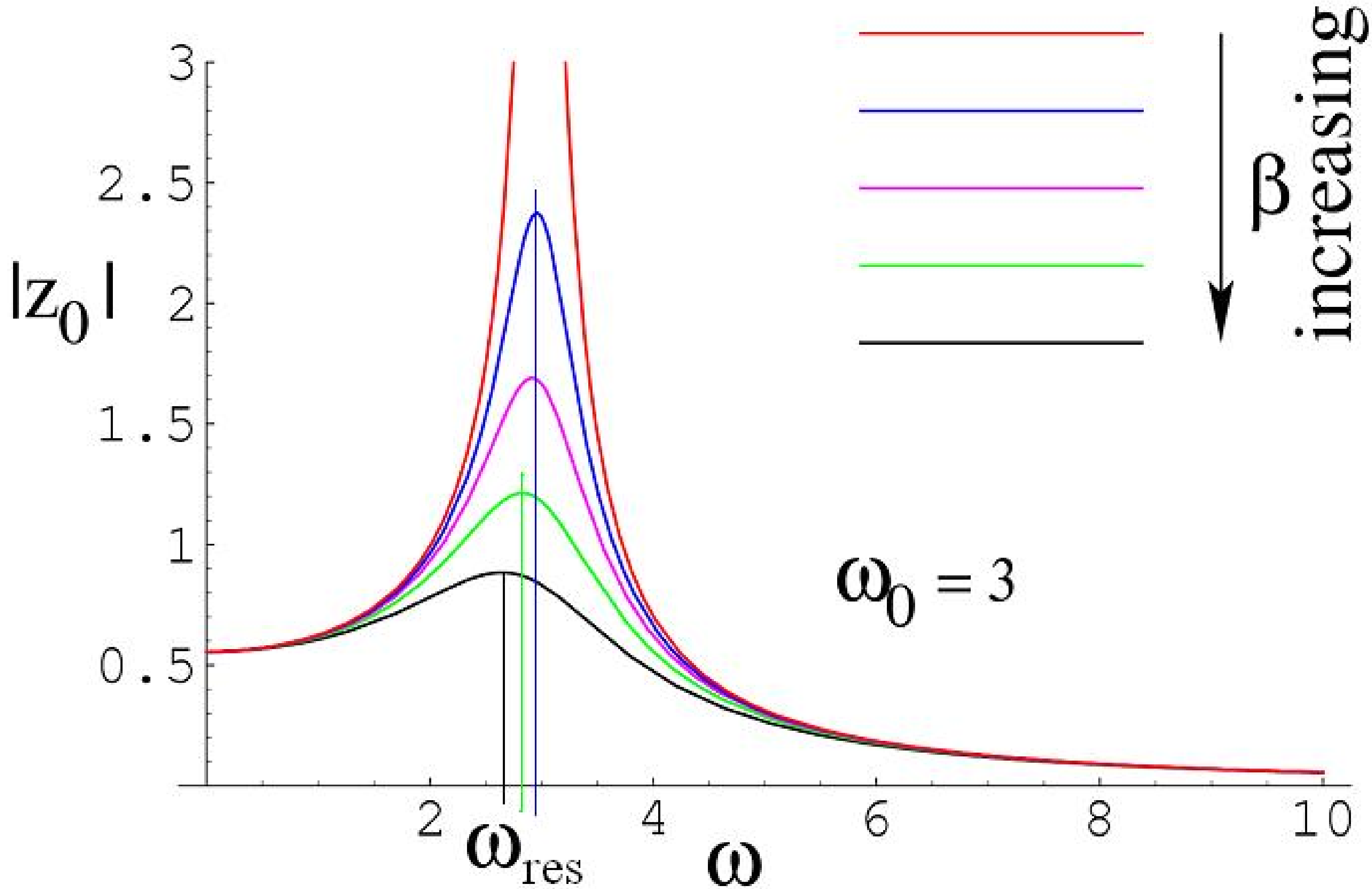
$f_0/2\beta\sqrt{\omega_0^2-\beta^2}$
1
0.8
$|z_0|$
0.6
0.4
0.2
$f_0/2\beta\omega_0$
ω_0=3
2
4
ω
6
8
10
ω_{res}

Phase:

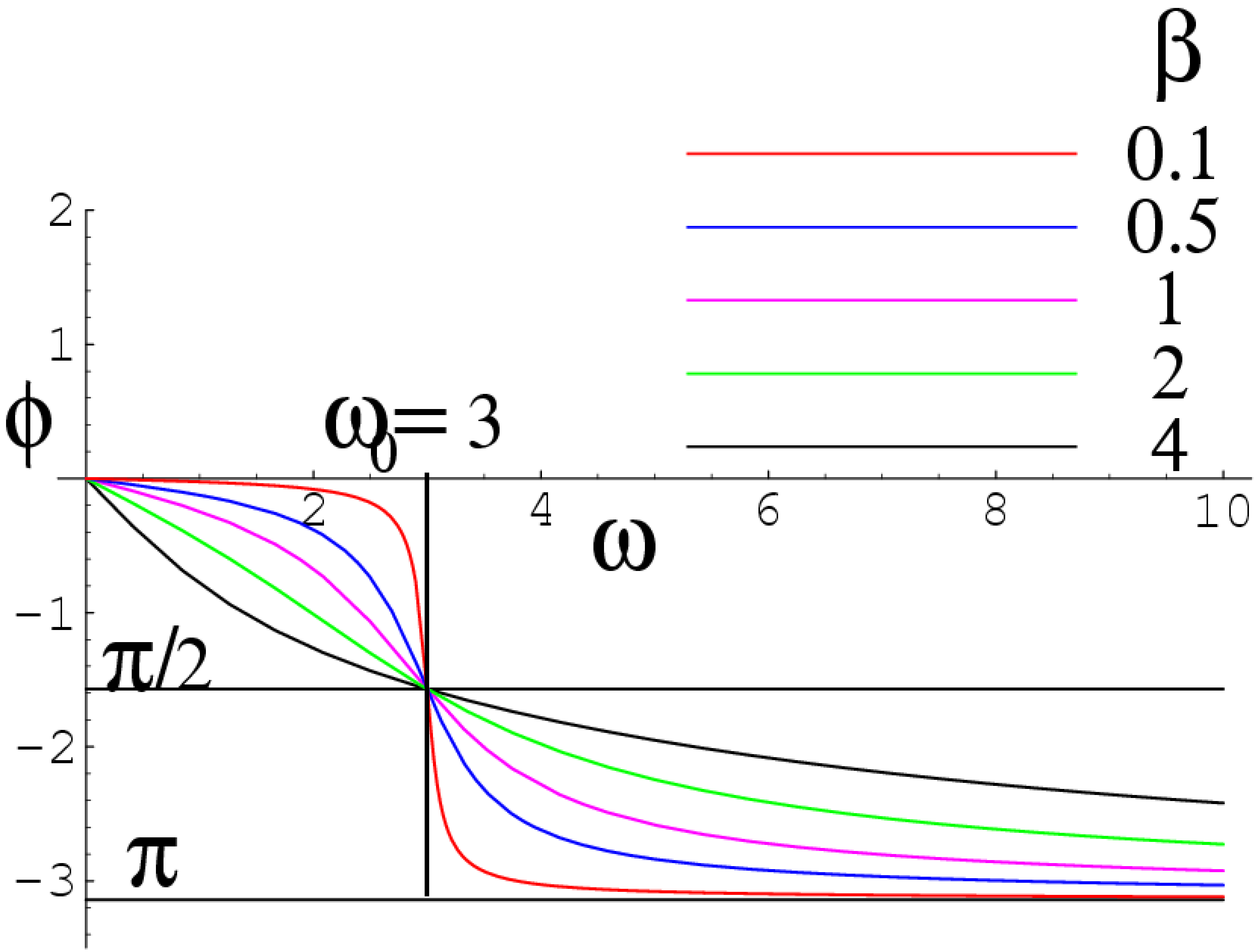




3
2.5
2
1.5
1
0.5
$|z_0|$
2
4
6
8
10
ω
β
increasing
$\omega_0 = 3$



β increasing
3
2.5
2
1.5
1
0.5
$|z_0|$
$\omega_0 = 3$
2
4
6
8
10
ω_{res}
ω

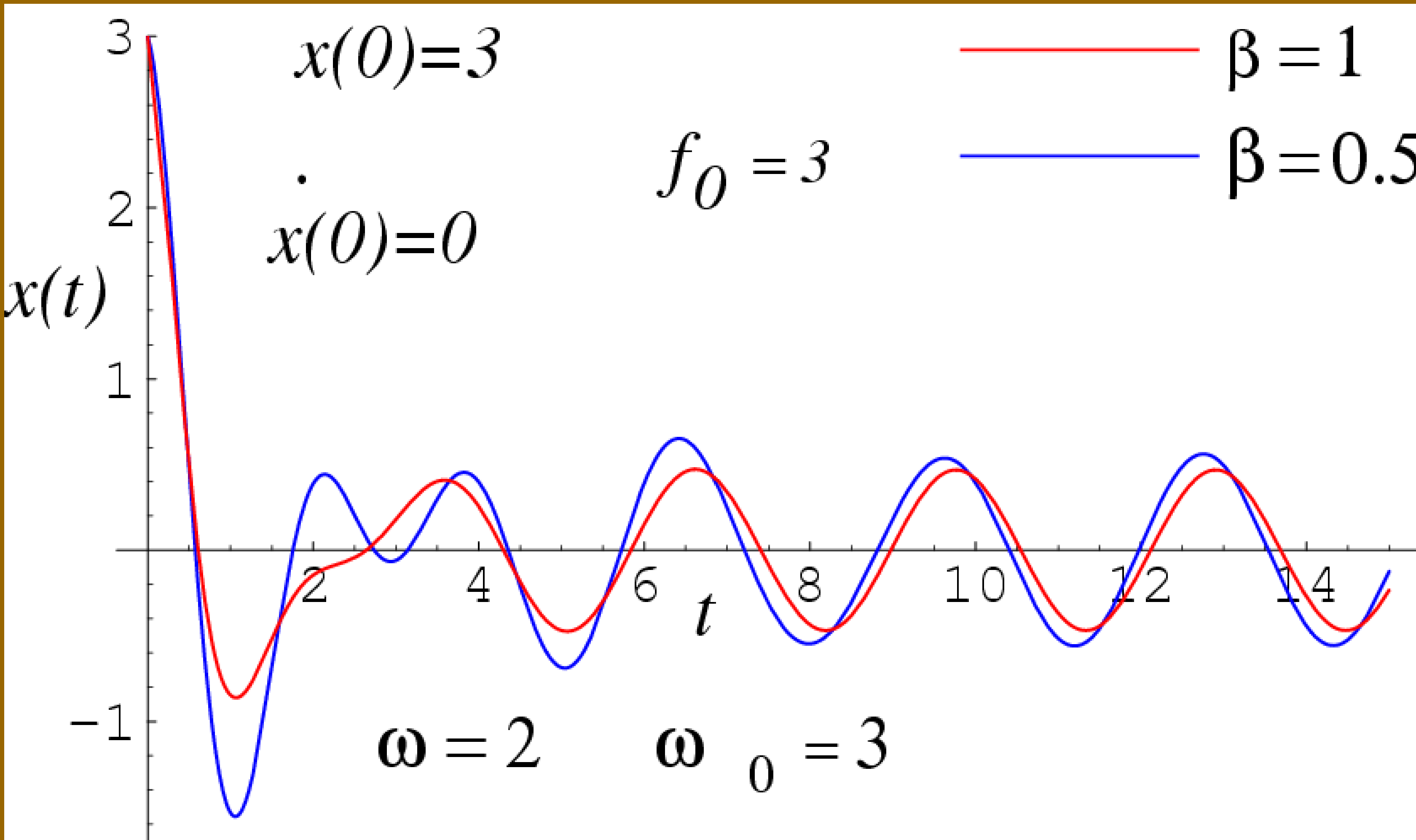

β
0.1
0.5
1
2
4
ϕ
ω₀ = 3
ω
π/2
π
2
1
−1
−2
−3
2
4
6
8
10

Show amplitude resonance at:

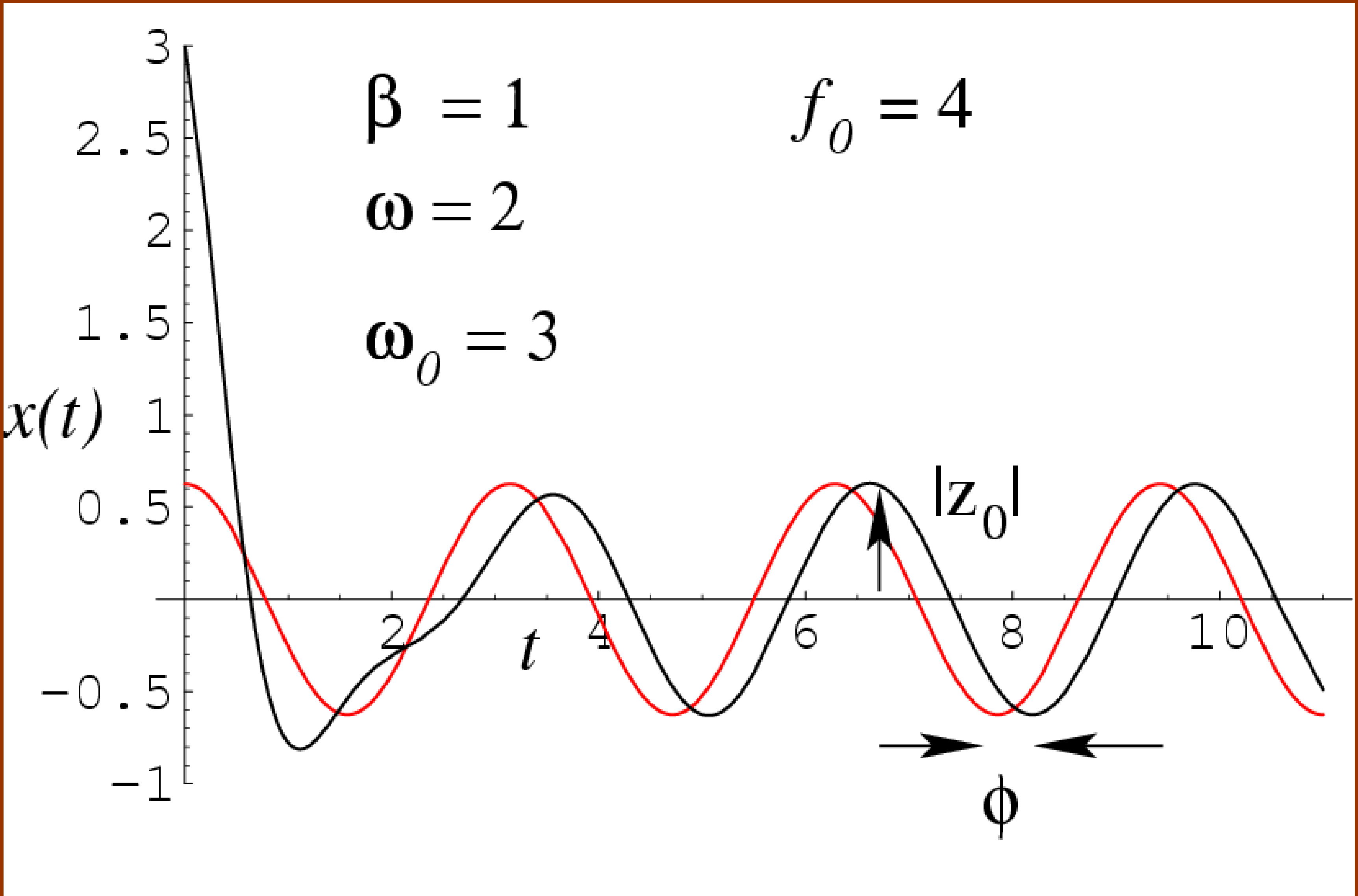
$$\omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$$

$$|z_0|_{res} = \frac{f_0}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

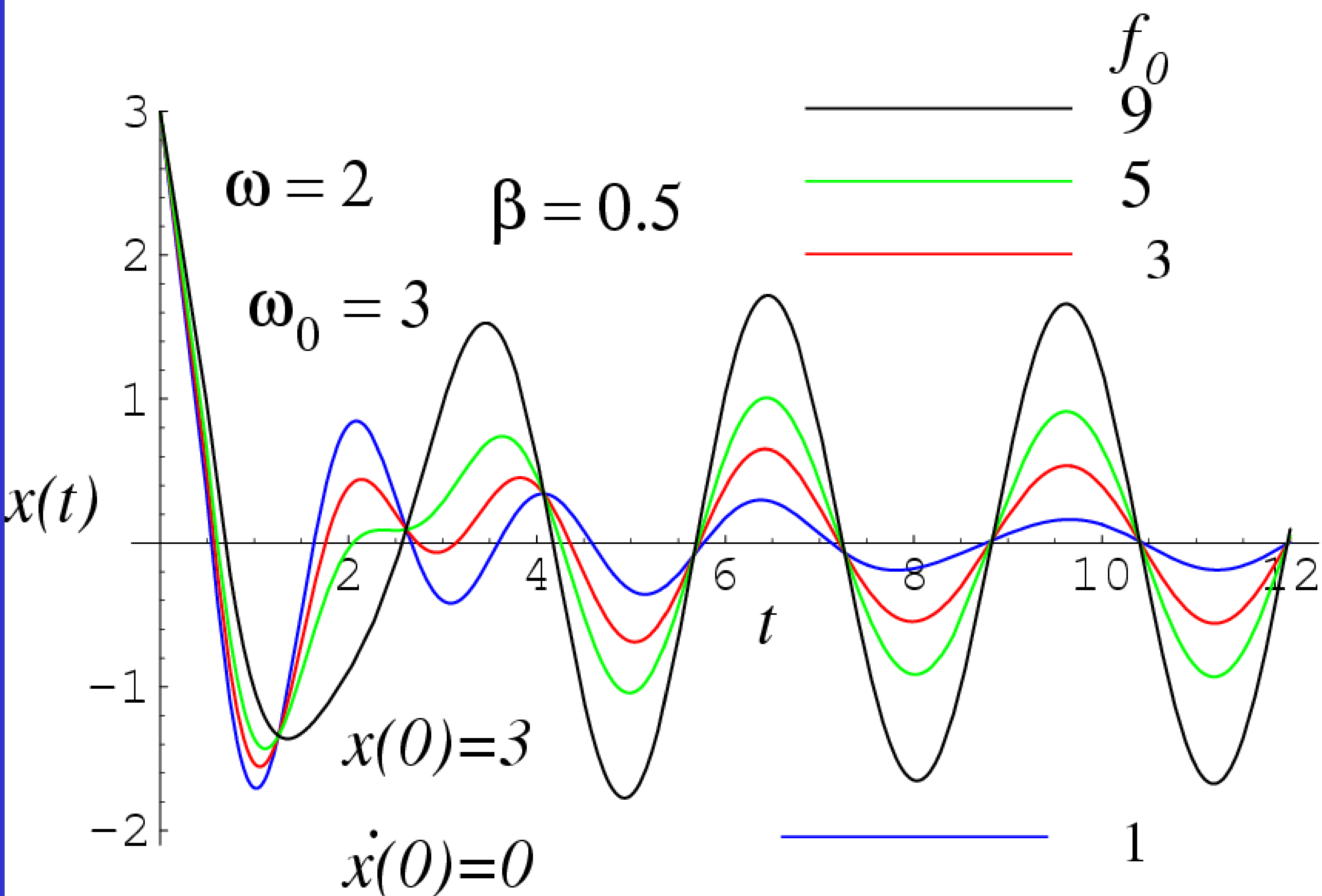
Forced oscillations for different resistances



Amplitude and phase



Different driving amplitudes



Case 1: **Low frequencies:** $\omega << \omega_0$

$$|z_0|_{st} \to \frac{f_0}{\omega_0^2} = \frac{F_0}{k}$$

$$\phi \to 0$$

$$x(t) \approx \frac{F_0}{k} \cos \omega t$$

Case 2: **High frequencies:** $\omega >> \omega_0$

$$|z_0| \rightarrow \frac{f_0}{\omega^2}$$

$$\phi \rightarrow -\pi$$

$$x(t) \approx -\frac{F_0}{m\omega^2}\cos\omega t$$

Case 3: **Intermediate frequencies:** $\omega \approx \omega_0$

For $\omega = \omega_0$

$$|z_0| = \frac{f_0}{2\beta\omega_0}$$

$$\phi \to -\frac{\pi}{2}$$

$$x(t) = \frac{F_0}{2m\beta\omega_0}\sin\omega_0 t \;\; = \frac{F_0}{2\,\varGamma\,\omega_0}\sin\omega_0 t$$

Construction of recording instruments:

$$F(t) = \sum_{n=0}^{+\infty}[C_n e^{in\omega t} + C_n^* e^{-in\omega t}]$$

$$C_n = (A_n - iB_n)/2$$

Highest frequency one would like to record (say) = N ω In practice

Sensitivity of the instrument: For N $\omega << \omega_0$

Can be enhanced by choosing small restoring force *k*

Distortion

Relative freedom from distortion

For forcing frequencies ≤ N ω

Amplitude $|z_0|$ ~ Almost constant

No marked resonance occurs

Choose $\omega_0^2 - 2\beta^2 = 0$

First choose ω_0 high enough by

choosing *k* and *m*, then choose *r* to satisfy

$$\omega_0^2 = 2\beta^2 = 100$$