Damped harmonic motion

Resistance is proportional to velocity.

$$F = m\ddot{x} = -2r\dot{x} - kx$$

$$m\ddot{x} + 2r\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2r}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Case I: Underdamped $(\beta^2 < \omega_0^2)$

$$x(t) = \exp(-\beta t)(A\cos(\omega' t) + B\sin(\omega' t))$$

$$\dot{x}(t) = \exp(-\beta t)\left[(-A\beta + B\omega')\cos(\omega' t) - (A\omega' + B\beta)\sin(\omega' t)\right]$$

Initial conditions

$$x(0) = x_0 \quad \Rightarrow \quad A = x_0$$

$$\dot{x}(0) = v_0 \quad \Rightarrow \quad B = (v_0 + \beta x_0)/\omega'$$

Angular frequency

$$\omega' = \sqrt{\frac{k}{m} - \frac{r^2}{m^2}} = \sqrt{\omega_0^2 - \beta^2}$$

Problem:

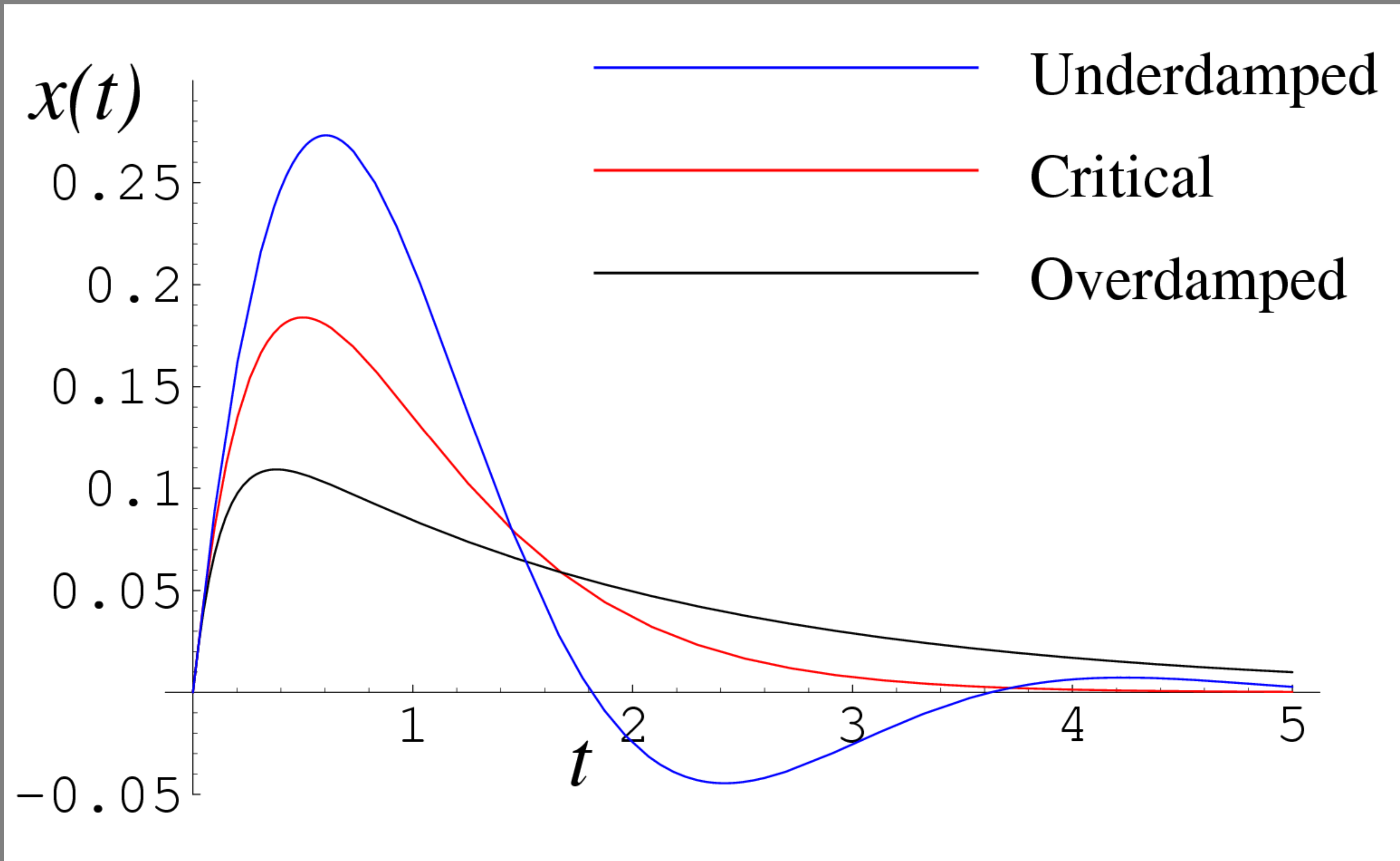
We have shown that the general solution x(t) with two constants can describe the motion of damped oscillator with any initial conditions. Show that there does not exist any other solution satisfying the same initial conditions.

CaseII: Critical damping $(\beta^2 = \omega_0^2)$

$$A \exp(-\beta t)$$

$$Bt \exp(-\beta t)$$

$$x(t) = (A + Bt) \exp(-\beta t)$$



x(t)
0.25
0.2
0.15
0.1
0.05
-0.05
1
2
3
4
5
t
Underdamped
Critical
Overdamped

Case III: Overdamped $(\beta^2 > \omega_0^2)$

$$x(t) = A_1 \exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\, t)$$
$$+ A_2 \exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\, t)$$

$$x(t) = A \exp(-\beta t) \cosh(\sqrt{\beta^2 - \omega_0^2}\, t)$$
$$+ B \exp(-\beta t) \sinh(\sqrt{\beta^2 - \omega_0^2}\, t)$$

$$A_1 = (A + B)/2 \qquad A_2 = (A - B)/2$$

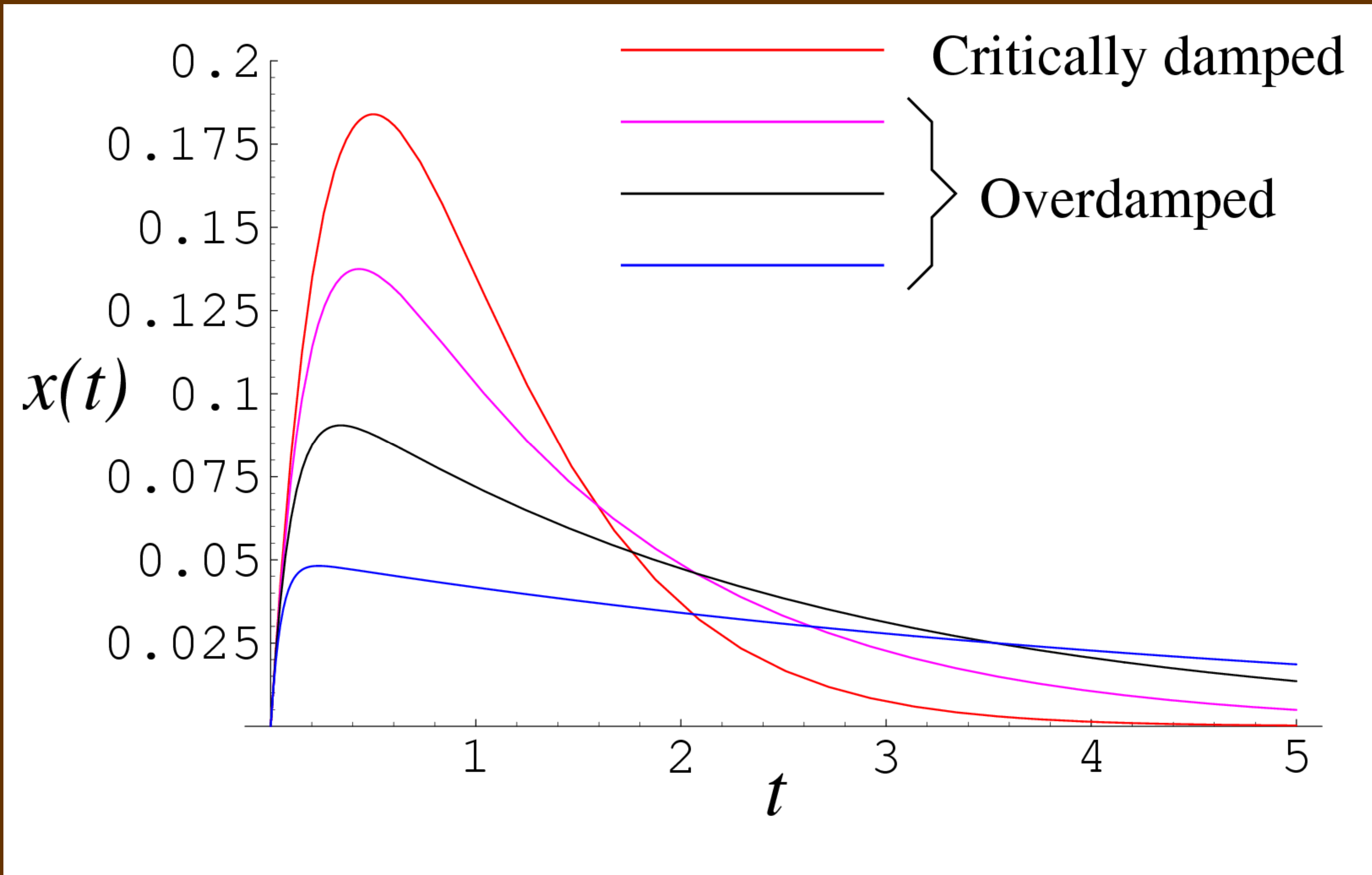
$$x(0) = A_1 + A_2 = A$$

$$\dot{x}(t) = A_1(-\beta+\sqrt{\beta^2 - \omega_0^2})\exp(-\beta t+\sqrt{\beta^2 - \omega_0^2}\,t)$$

$$+A_2(-\beta - \sqrt{\beta^2 - \omega_0^2})\exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\,t)$$

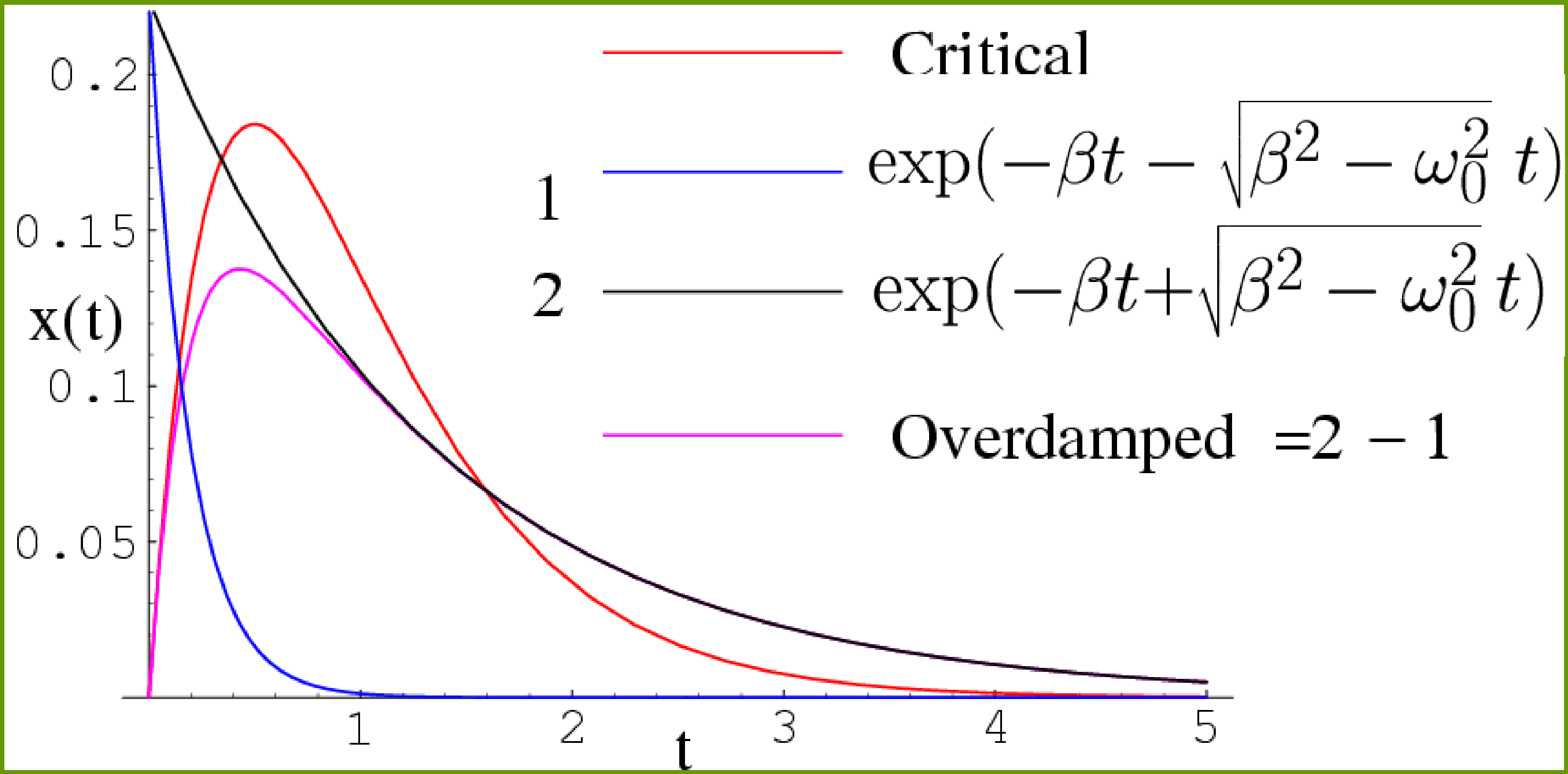
$$\dot{x}(0) = -\beta(A_1 + A_2) + \sqrt{\beta^2 - \omega_0^2}\,(A_1 - A_2)$$

$$\frac{\dot{x}(0) + \beta x(0)}{\sqrt{\beta^2 - \omega_0^2}} = (A_1 - A_2) = B$$



Critically damped
Overdamped
x(t)
0.2
0.175
0.15
0.125
0.1
0.075
0.05
0.025
1
2
3
4
5
t

A=0   B=2



Critical
1
$\exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\; t)$
2
$\exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\; t)$
Overdamped =2 − 1
x(t)
t
0.2
0.15
0.1
0.05
1
2
3
4
5

Forced or driven oscillator:

Undamped case:

$$\begin{aligned} m\ddot{x} + kx &= F(t) \\ \ddot{x} + \omega_0^2 x &= F(t)/m \end{aligned}$$

Particular solution: P(t)

$$\frac{d^2P(t)}{dt^2} + \omega_0^2 P(t) = F(t)/m$$

Complementary function: C(t)

$$\frac{d^2C(t)}{dt^2} + \omega_0^2 C(t) = 0$$

$$\frac{d^2(P(t) + C(t))}{dt^2} + \omega_0^2(P(t) + C(t)) = F(t)/m$$

General solution:

$$x(t) = P(t) + C(t)$$

Sinusoidal forcing:

$$F(t) = F_0 \sin \omega t$$

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \sin \omega t = f_0 \sin \omega t$$

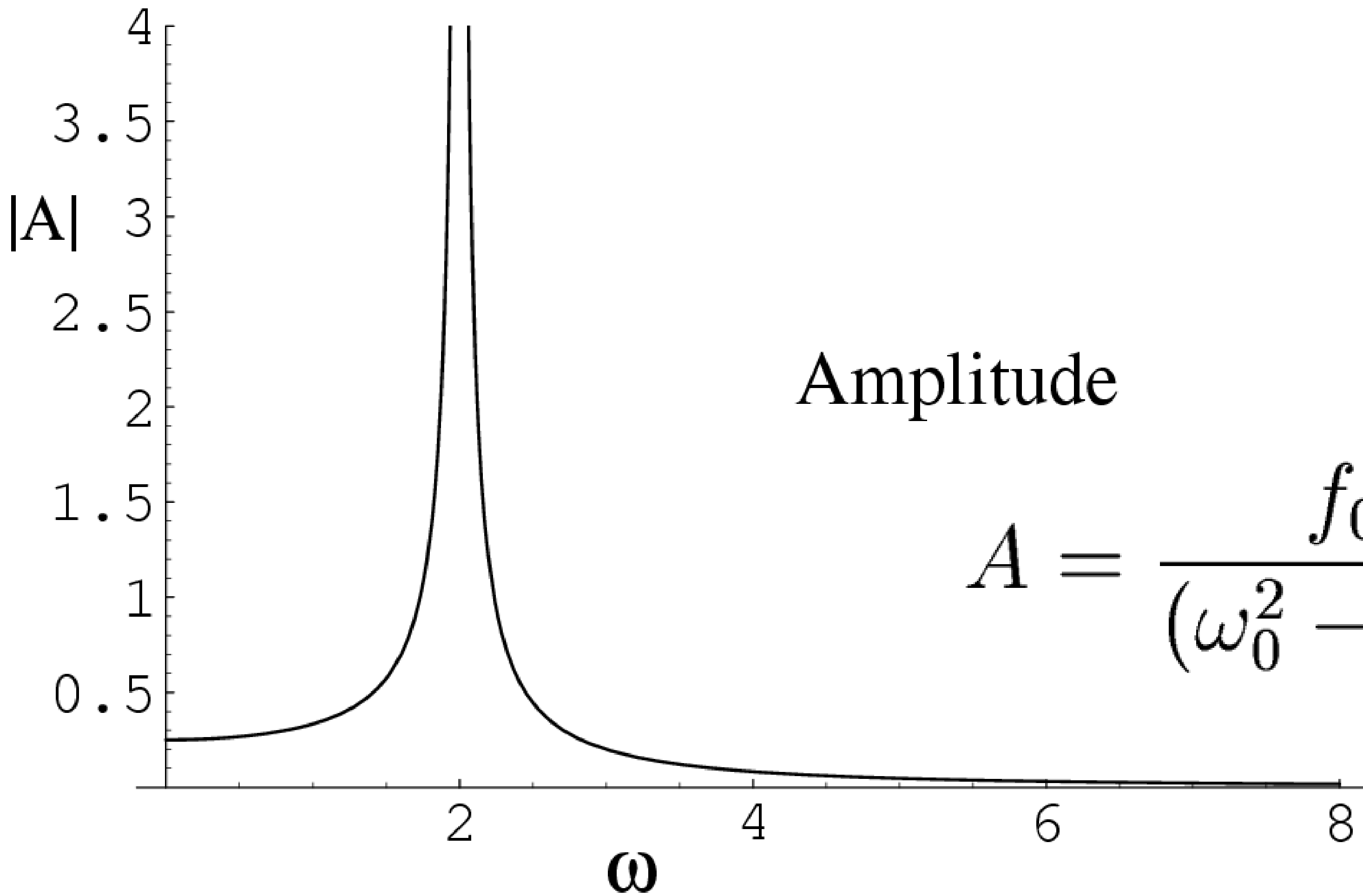
Trial solution: $P(t) = A \sin \omega t$

$$-A\omega^2 \sin \omega t + A\omega_0^2 \sin \omega t = f_0 \sin \omega t.$$

$$A = \frac{f_0}{(\omega_0^2 - \omega^2)}$$

General solution:

$$x(t) = \frac{f_0}{\omega_0^2 - \omega^2}\sin\omega t + B\cos\omega_0 t + C\sin\omega_0 t.$$

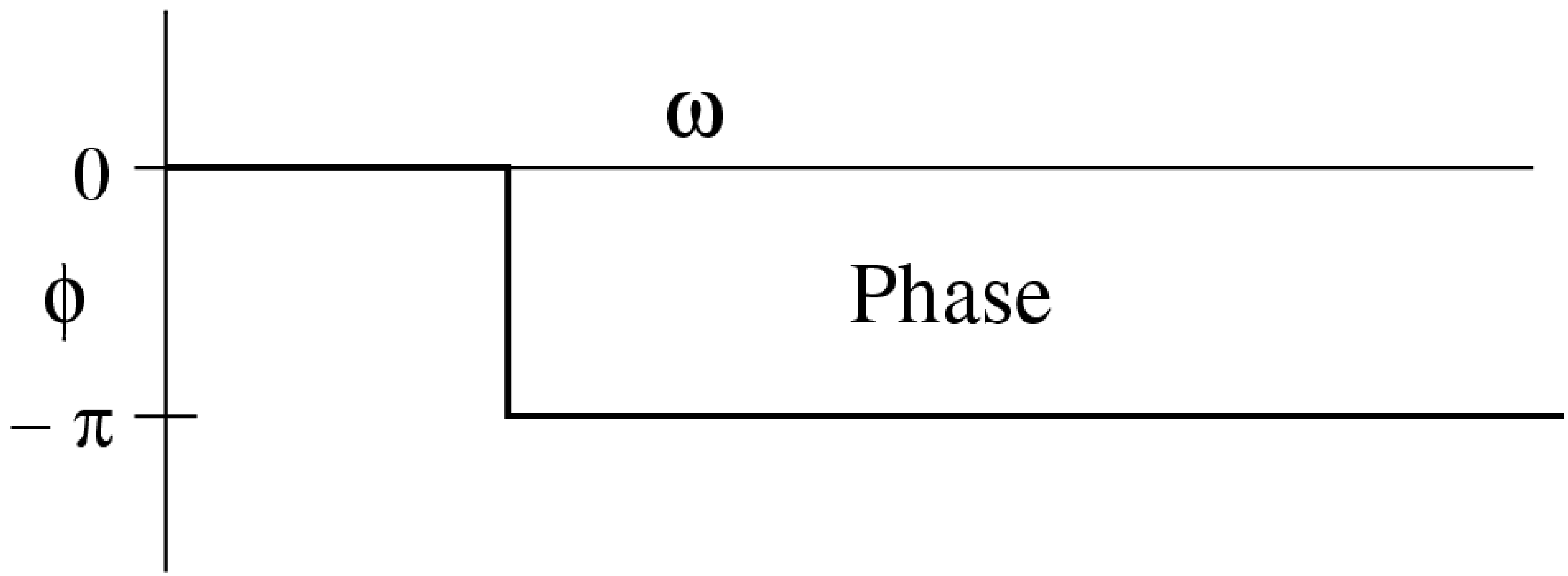


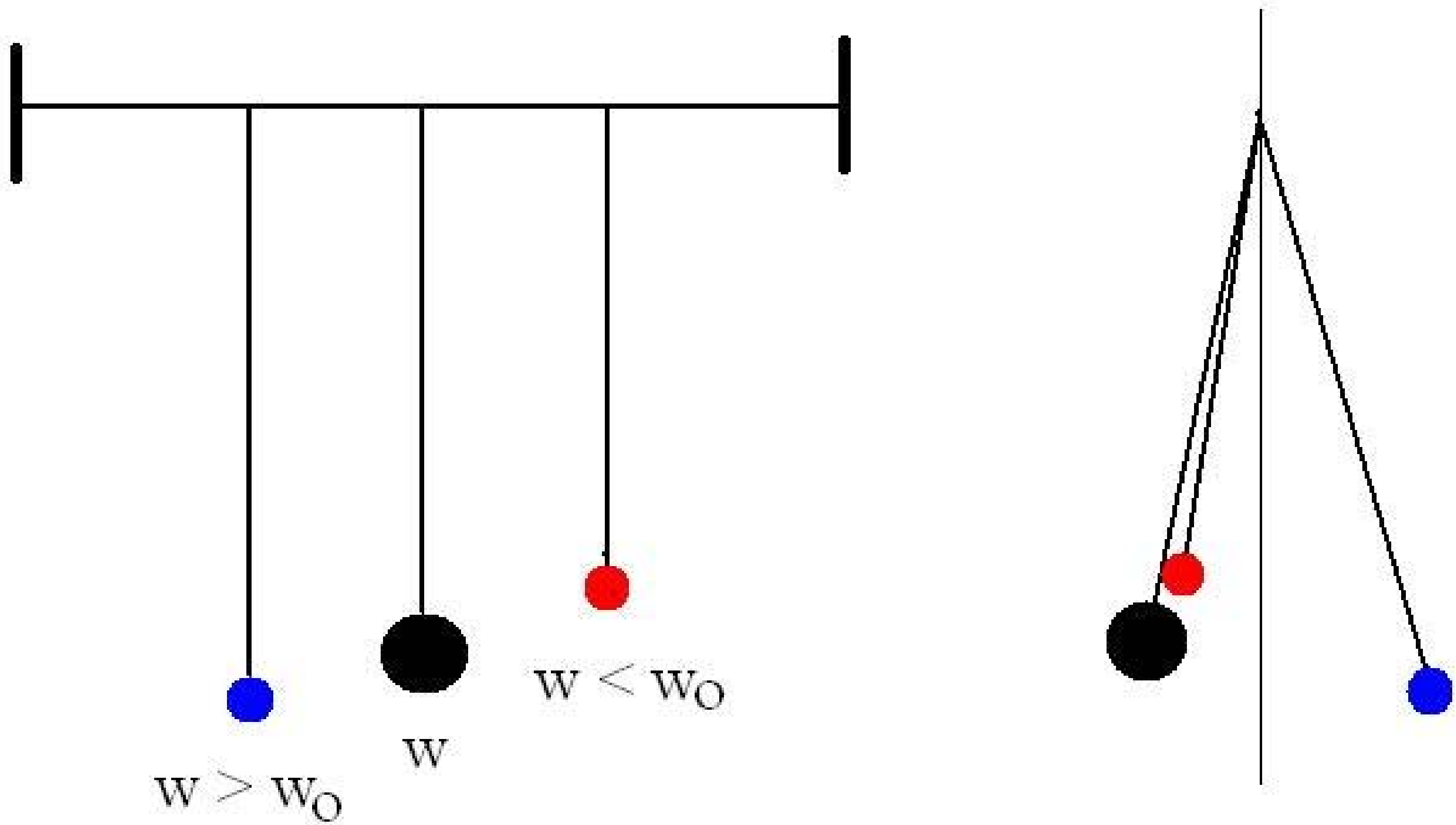
Amplitude

$$A = \frac{f_0}{(\omega_0^2 - \omega^2)}$$

Phase

$$x(t) = \frac{f_0}{\omega^2 - \omega_0^2} \sin(\omega t - \pi), \qquad (\omega > \omega_0)$$




w > w0
w
w < w0

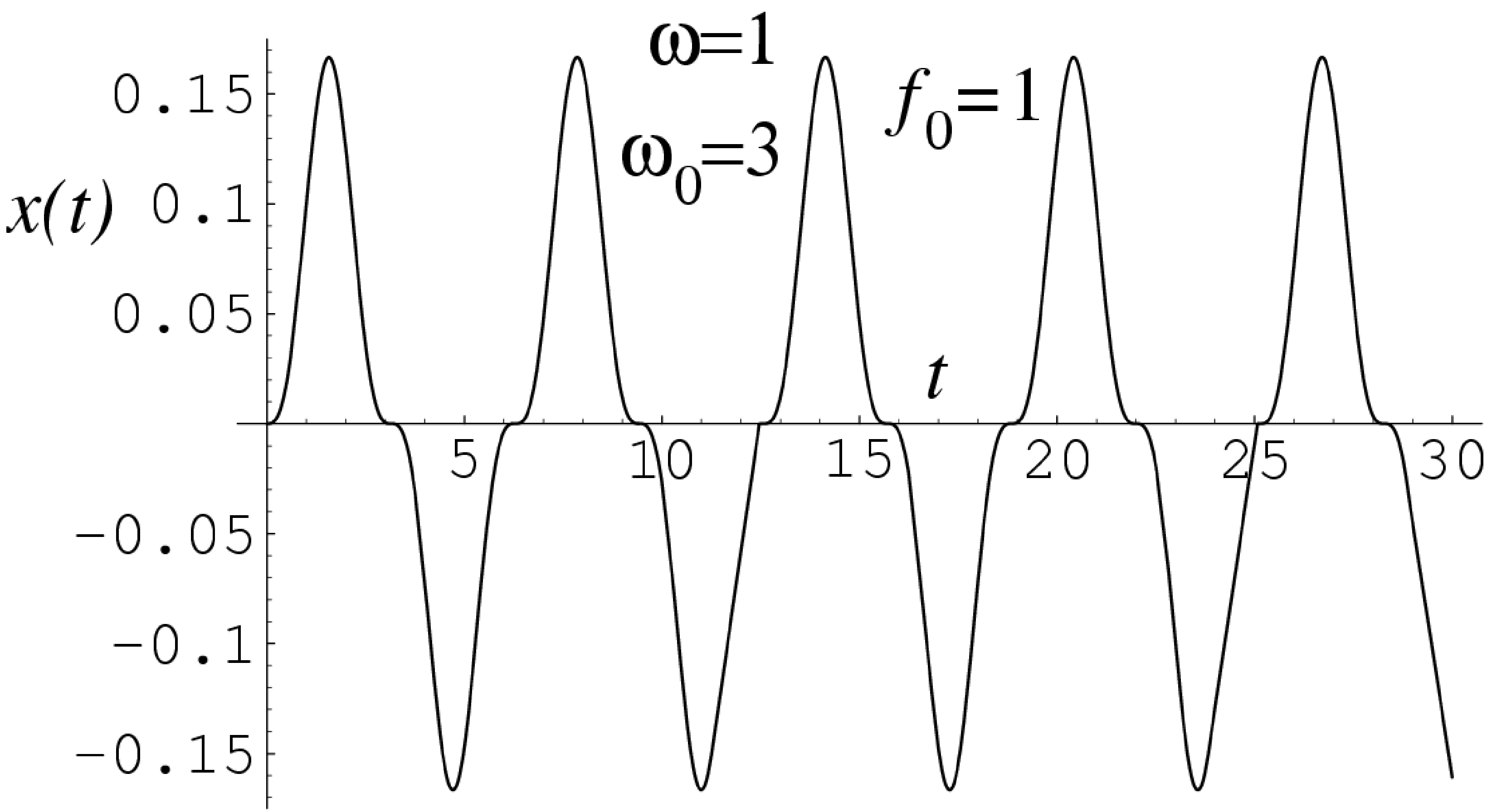
$$x(t=0)=\dot{x}(t=0)=0$$

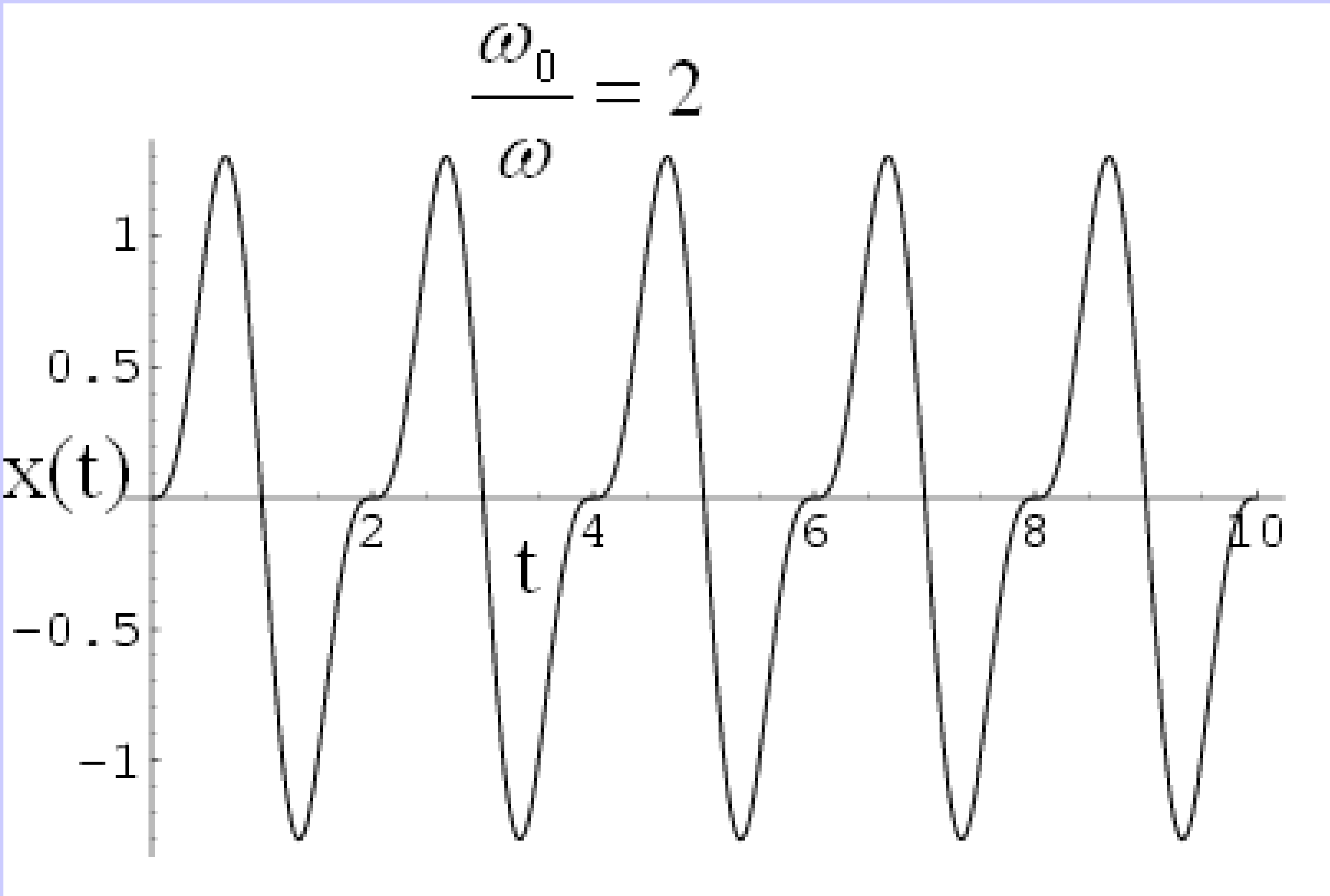
$$B=0$$

$$C=-\frac{f_0\omega}{\omega_0(\omega_0^2-\omega^2)}=-A\omega/\omega_0$$

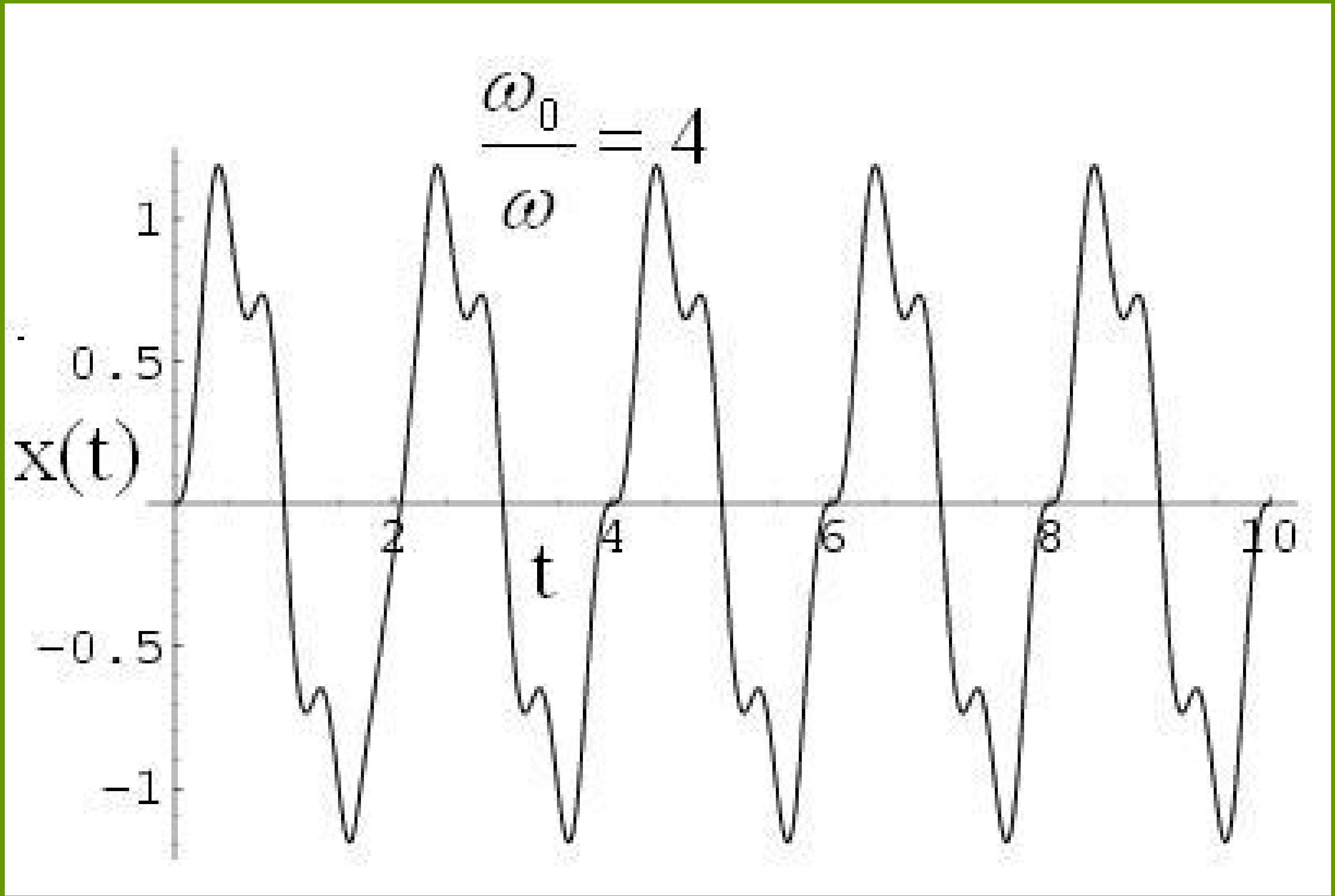
$$x(t)=A(\sin\omega t-\frac{\omega}{\omega_0}\sin\omega_0 t)$$

$$x(t=0) = \dot{x}(t=0) = 0$$

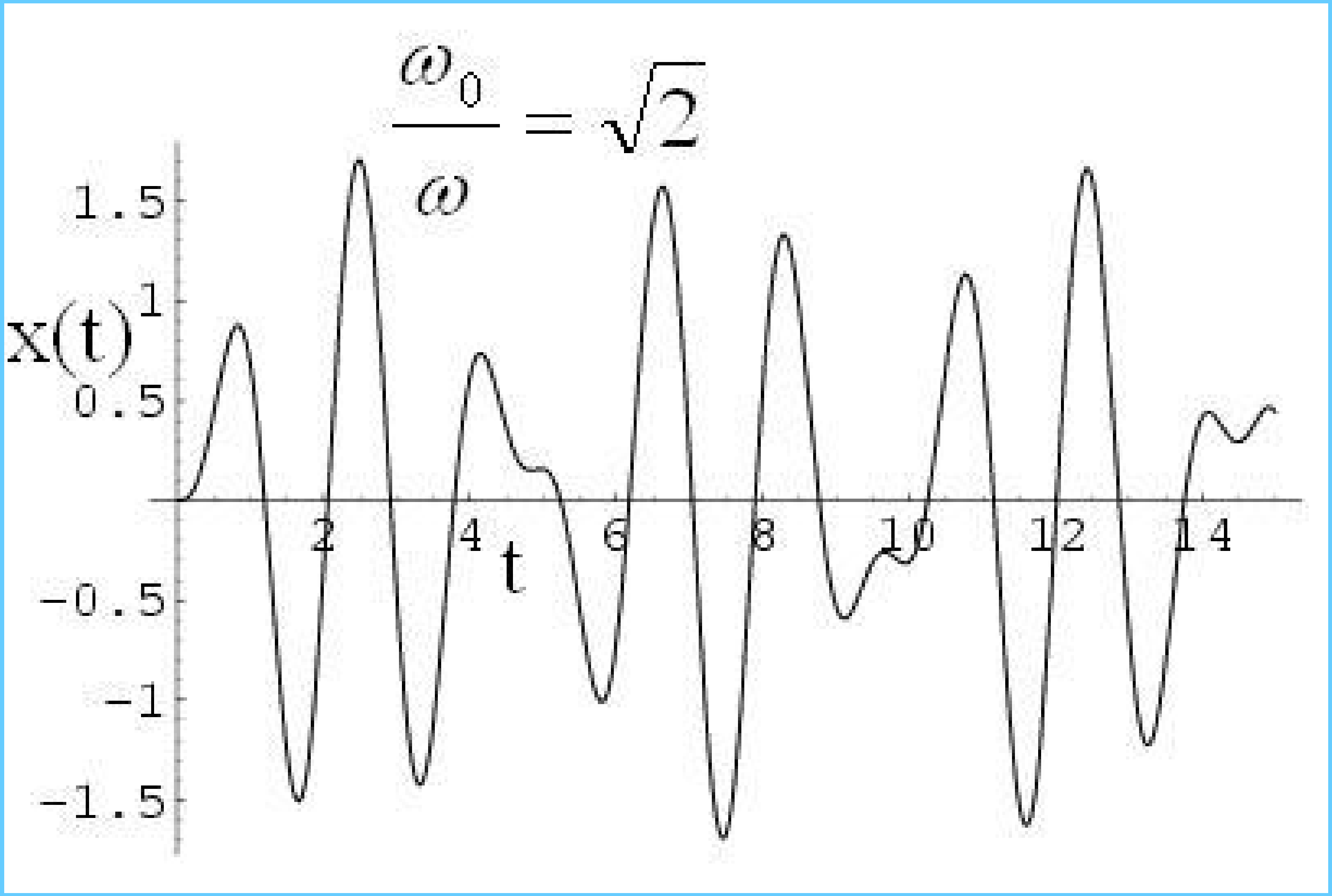




$\frac{\omega_0}{\omega} = 2$
x(t)
t
1
0.5
-0.5
-1
2
4
6
8
10



$\frac{\omega_0}{\omega} = 4$
1
0.5
x(t)
-0.5
-1
2
4
6
8
10
t


$\frac{\omega_0}{\omega} = \sqrt{2}$
x(t)
1.5
1
0.5
-0.5
-1
-1.5
2
4
6
8
10
12
14
t