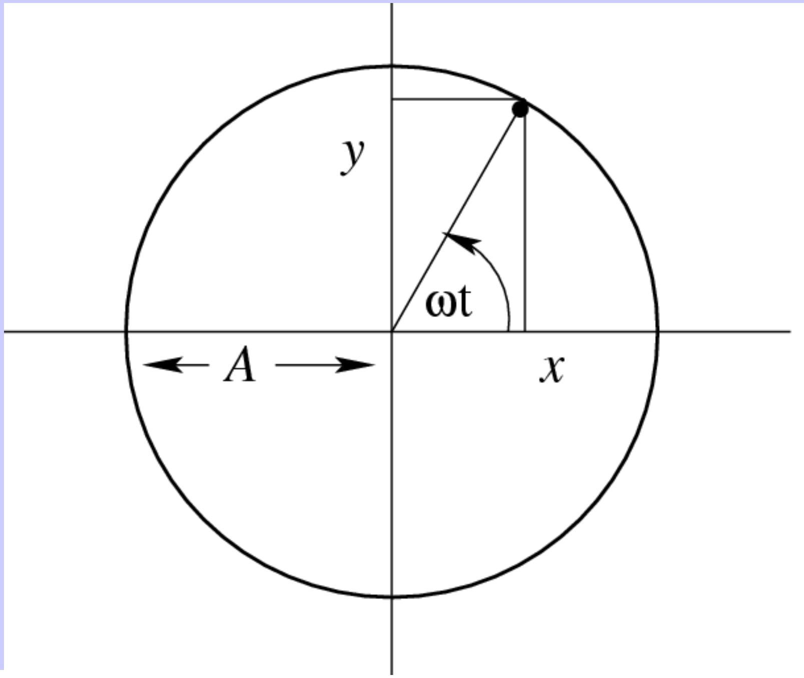
Uniform circular motion and SHM:



$$x = A\cos(\omega t) \qquad \text{and} \qquad y = A\sin(\omega t).$$

$$x = A\cos(\omega t + \phi) \qquad \text{and} \qquad y = A\sin(\omega t + \phi).$$

Superposition of two SHM in perpendicular directions:

$$x(t) = a\cos(\omega_1 t + \phi_1)$$
$$y(t) = b\cos(\omega_2 t + \phi_2)$$

Case I: $\omega_{\mathbf{1}} = \omega_{\mathbf{2}} = \omega$

$$\left(\frac{x}{a}\cos\phi_2 - \frac{y}{b}\cos\phi_1\right) = \sin(\omega t)\sin(\phi_2 - \phi_1)$$

$$\left(\frac{x}{a}\sin\phi_2 - \frac{y}{b}\sin\phi_1\right) = \cos(\omega t)\sin(\phi_2 - \phi_1).$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1)$$

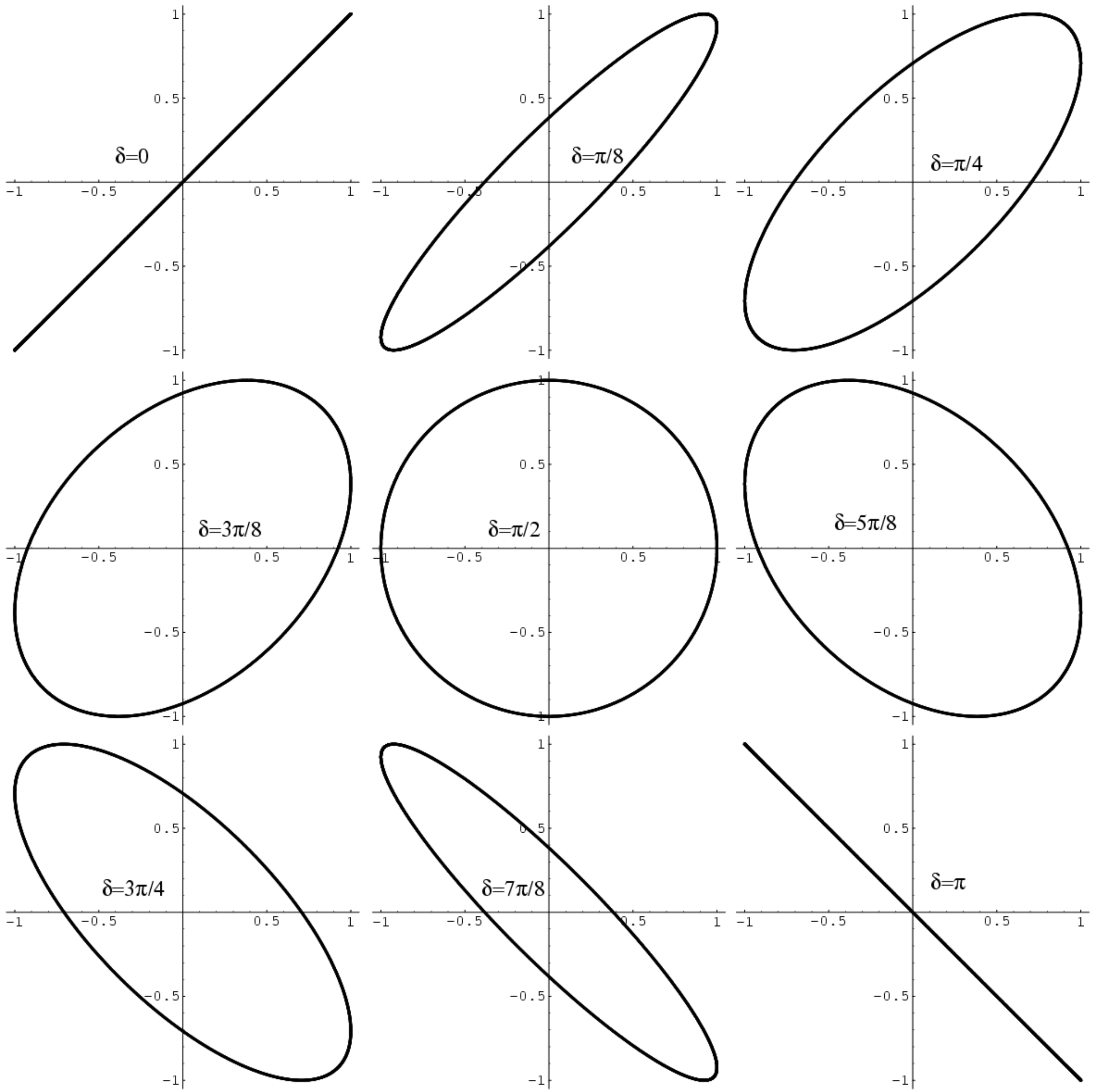
$$\delta = \phi_2 - \phi_1 = \pm n\pi$$

Straight line

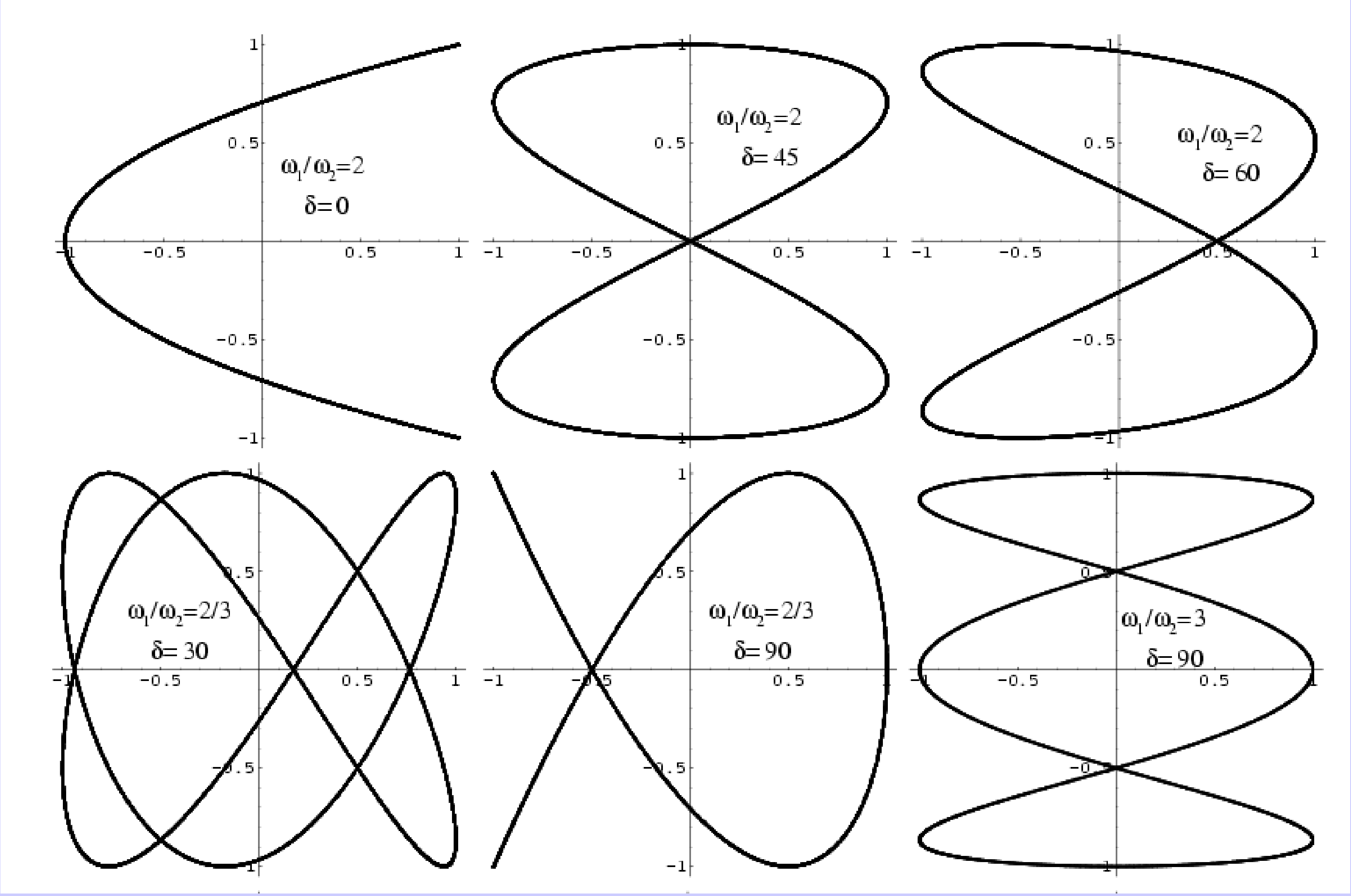
$$\delta = \pi/2 \text{ and } a = b$$

Circle

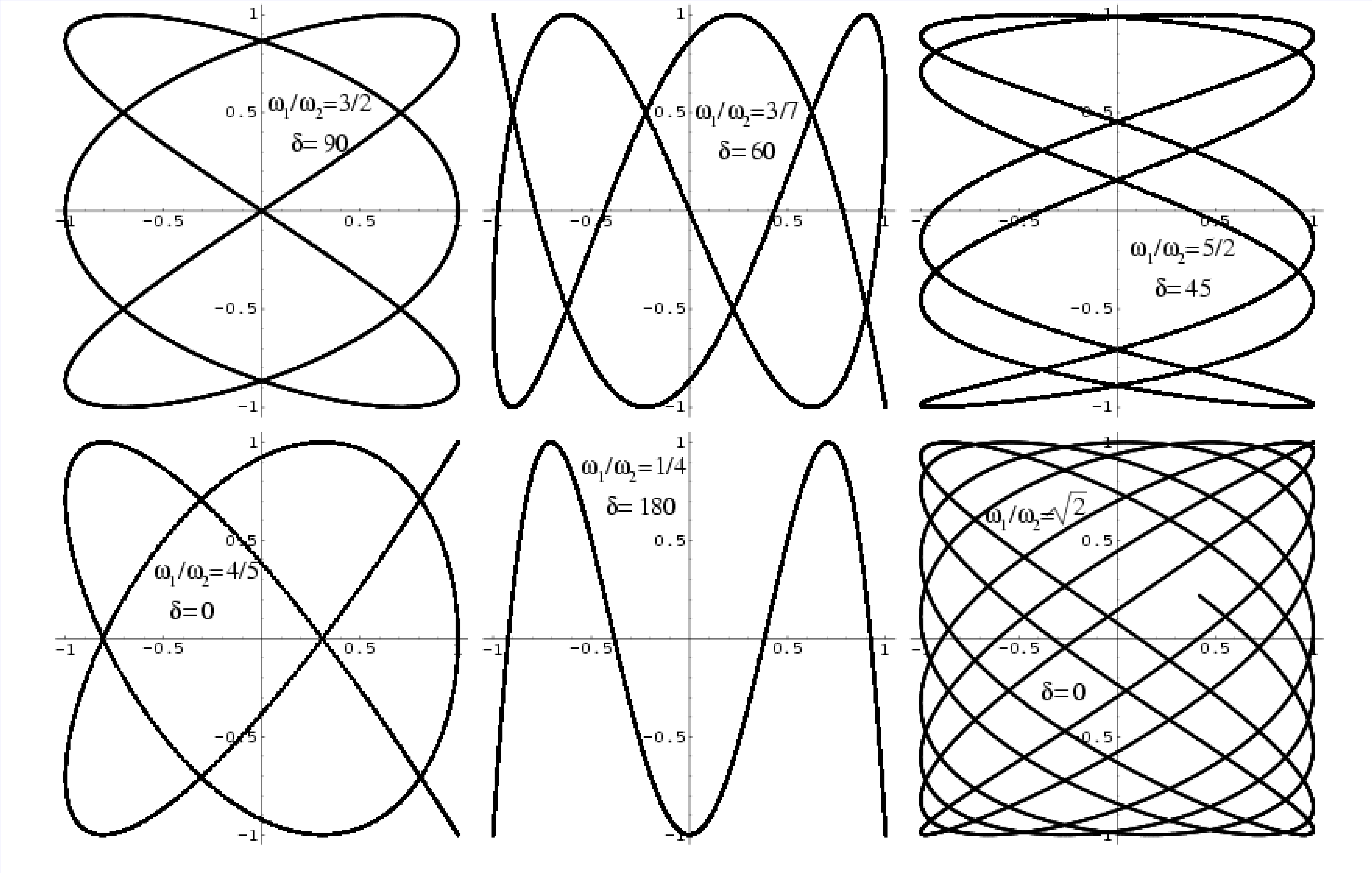
Ellipse otherwise


δ=0
δ=π/8
δ=π/4
δ=3π/8
δ=π/2
δ=5π/8
δ=3π/4
δ=7π/8
δ=π

Case II: $\omega_1 \neq \omega_2$



Lissajous figures



Hyperbolic functions

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}$$

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d\cosh x}{dx} = \sinh x \qquad \frac{d\sinh x}{dx} = \cosh x$$

$$\cosh ix = \frac{\exp(ix) + \exp(-ix)}{2} = \cos x$$

$$\sinh ix = \frac{\exp(ix) - \exp(-ix)}{2} = i\sin x$$

Damped harmonic motion

Resistance is proportional to velocity.

$$F = m\ddot{x} = -2r\dot{x} - kx$$

$$m\ddot{x} + 2r\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2r}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Try solution

$$x = A\exp(\gamma t)$$

$$A\gamma^2 \exp(\gamma t) + 2\beta A\gamma \exp(\gamma t) + \omega_0^2 A \exp(\gamma t) = 0$$

$$\gamma^2 + 2\beta\gamma + \omega_0^2 = 0$$

$$\gamma = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Case I: Underdamped $(\beta^2 < \omega_0^2)$

$$x(t) = \exp(-\beta t)[A_1 \exp(-i\sqrt{\omega_0^2 - \beta^2}\, t)$$

$$+ A_2 \exp(i\sqrt{\omega_0^2 - \beta^2}\, t)]$$

$$A_1 = A_2 = A/2$$

$$x(t) = A \exp(-\beta t) \cos(\omega' t)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{r^2}{m^2}} = \sqrt{\omega_0^2 - \beta^2}$$

x(t) is real for all time: $x(t) = x(t)^*$

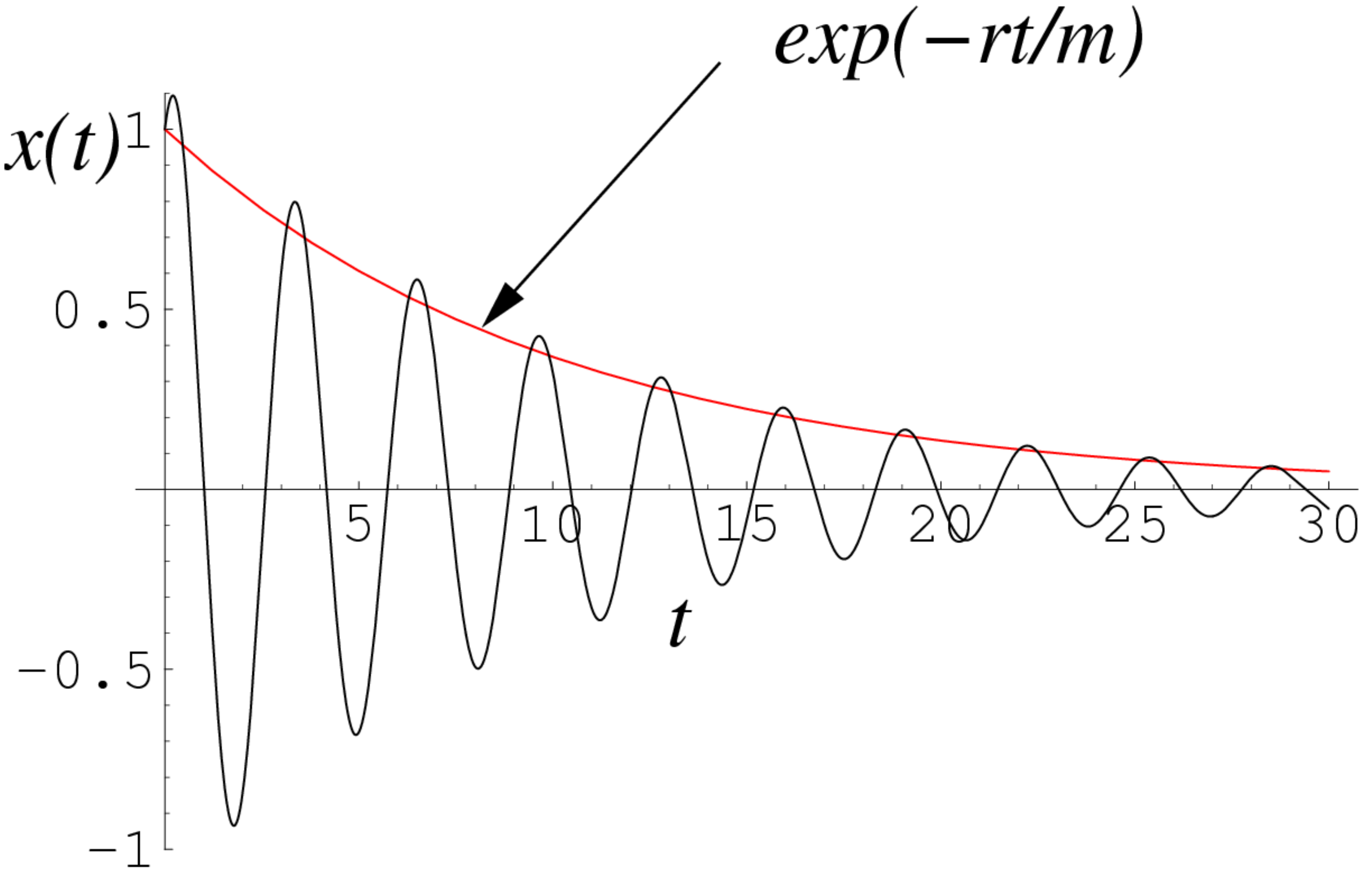
$$A_1^* = A_2 \qquad A_2^* = A_1$$

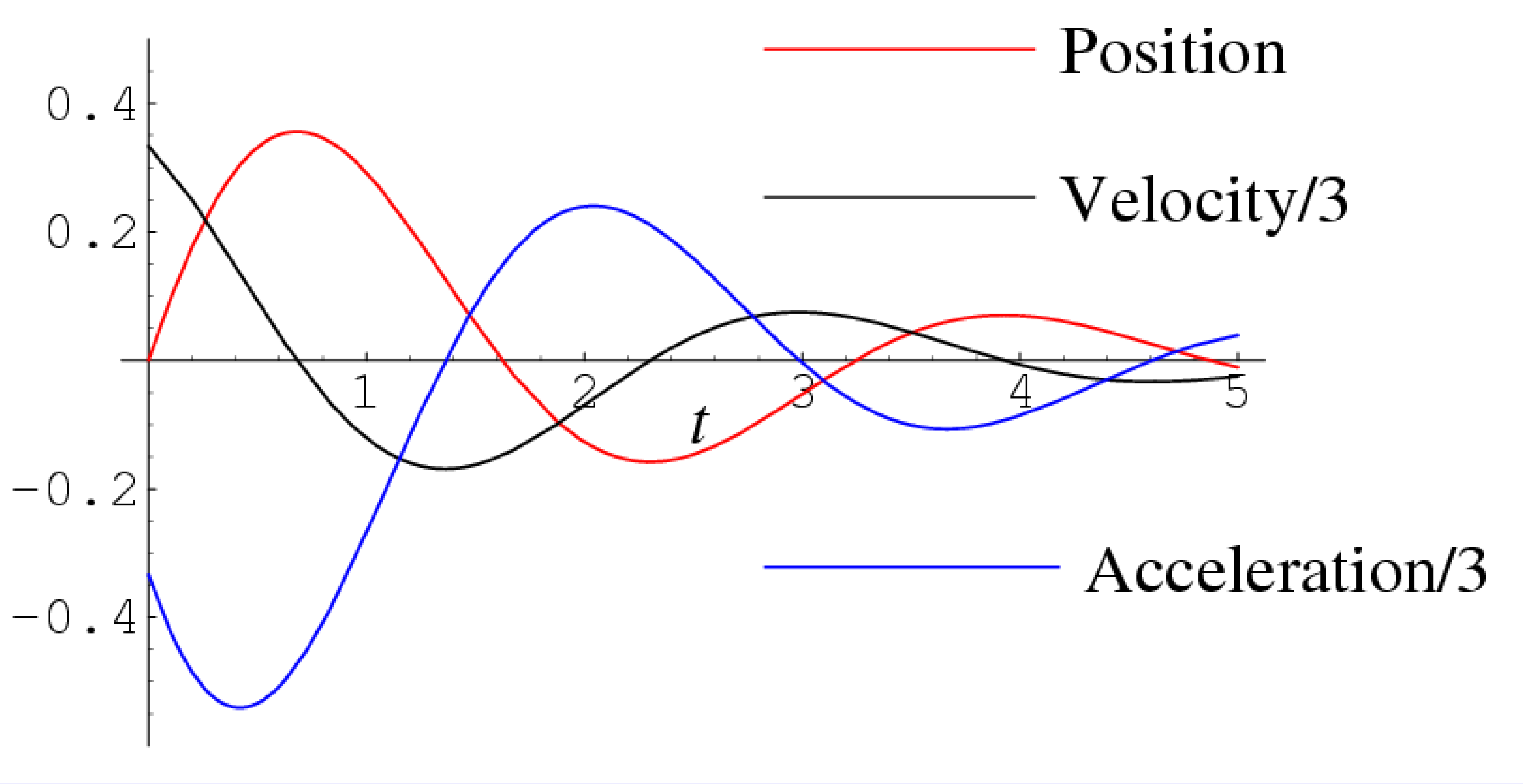
$$A_1 = \frac{A + iB}{2} \qquad A_2 = \frac{A - iB}{2}$$

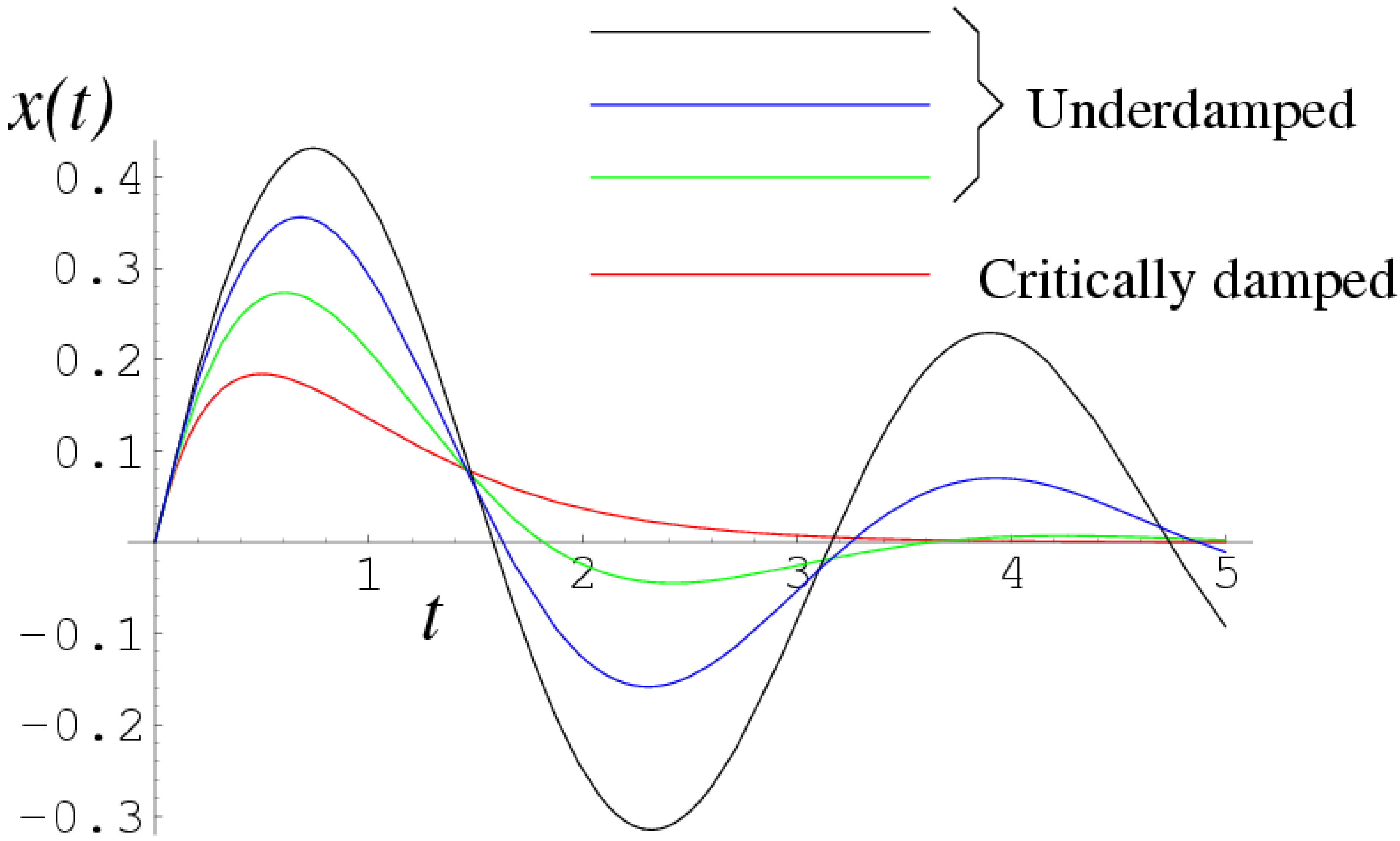
$$x(t) = \exp(-\beta t)(A\cos(\omega' t) + B\sin(\omega' t))$$

Angular frequency $\omega' = \sqrt{\omega_0^2 - \beta^2}$

Underdamped oscillations



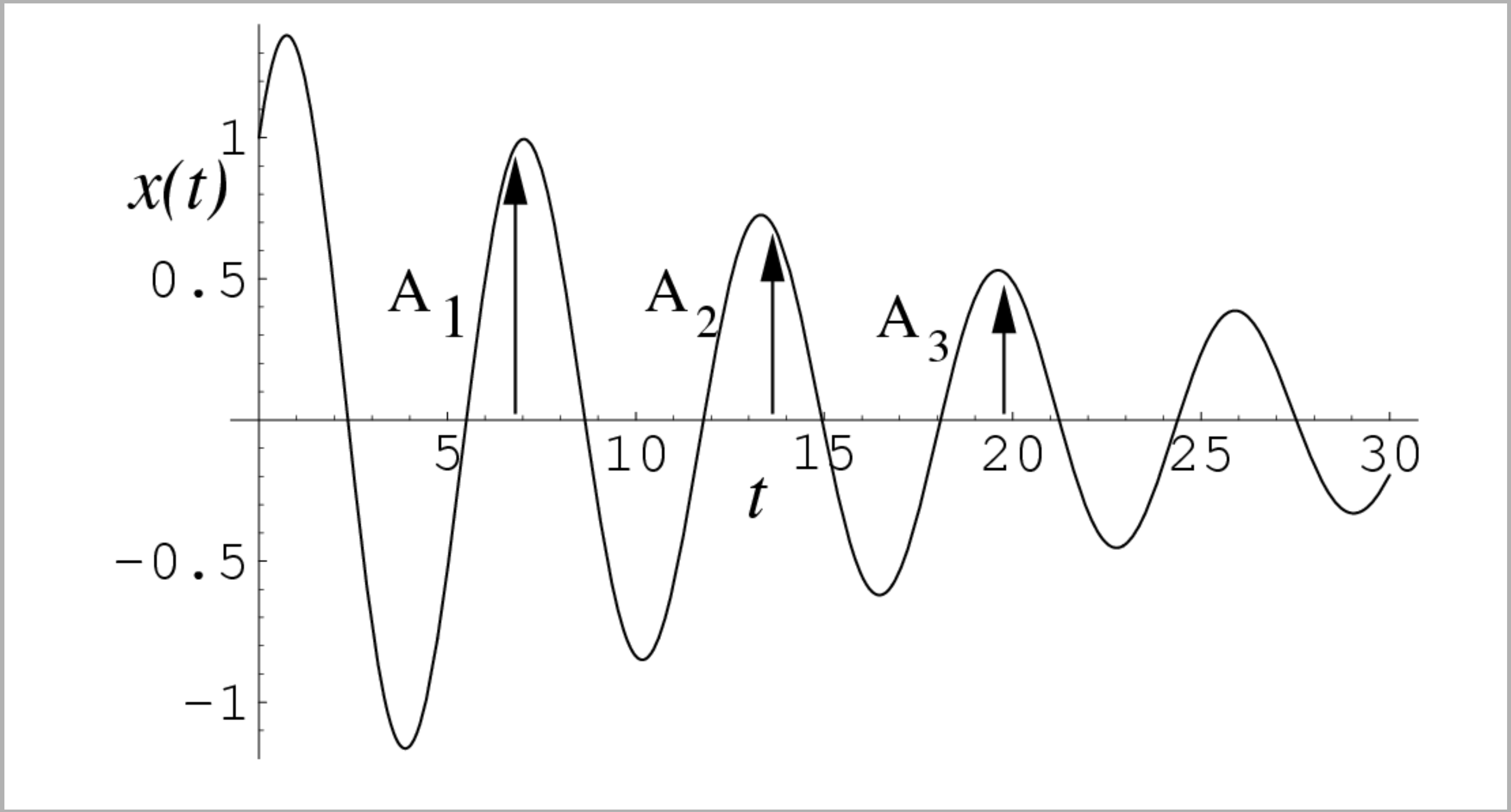

Position
Velocity/3
Acceleration/3
0.4
0.2
−0.2
−0.4
1
2
3
4
5
t



Underdamped
Critically damped
x(t)
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
1
2
3
4
5
t

Logarithmic decrement

$$\delta = \ln(A_n/A_{n+1}) = \beta T$$



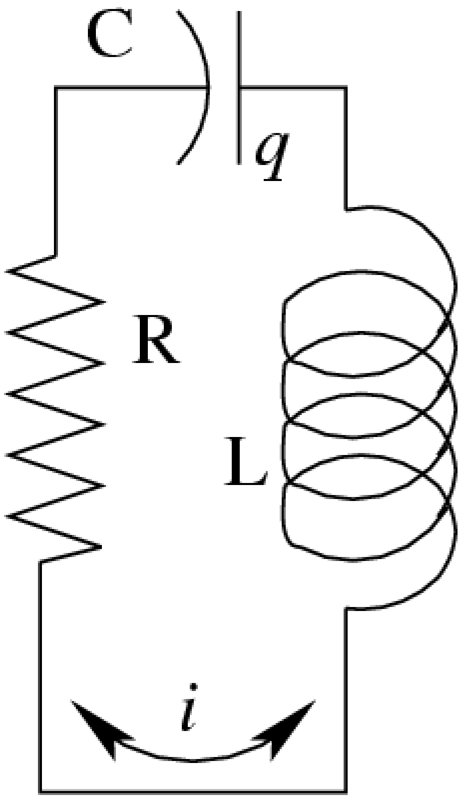
Relaxation time: $$\tau = \frac{1}{\beta} = \frac{m}{r}$$

Quality factor: $$Q = \frac{\omega'}{2\beta} = \frac{\pi}{\delta} = \frac{\omega' m}{2r}$$

Energy stored: $$E(t) = \frac{1}{2}\exp(-2\beta t)kA^2$$

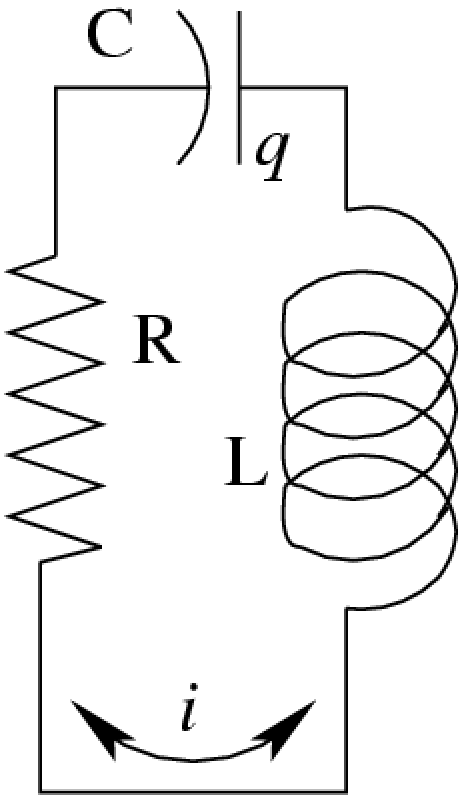
Example: LCR in series

Find charge on the capacitor at time t.



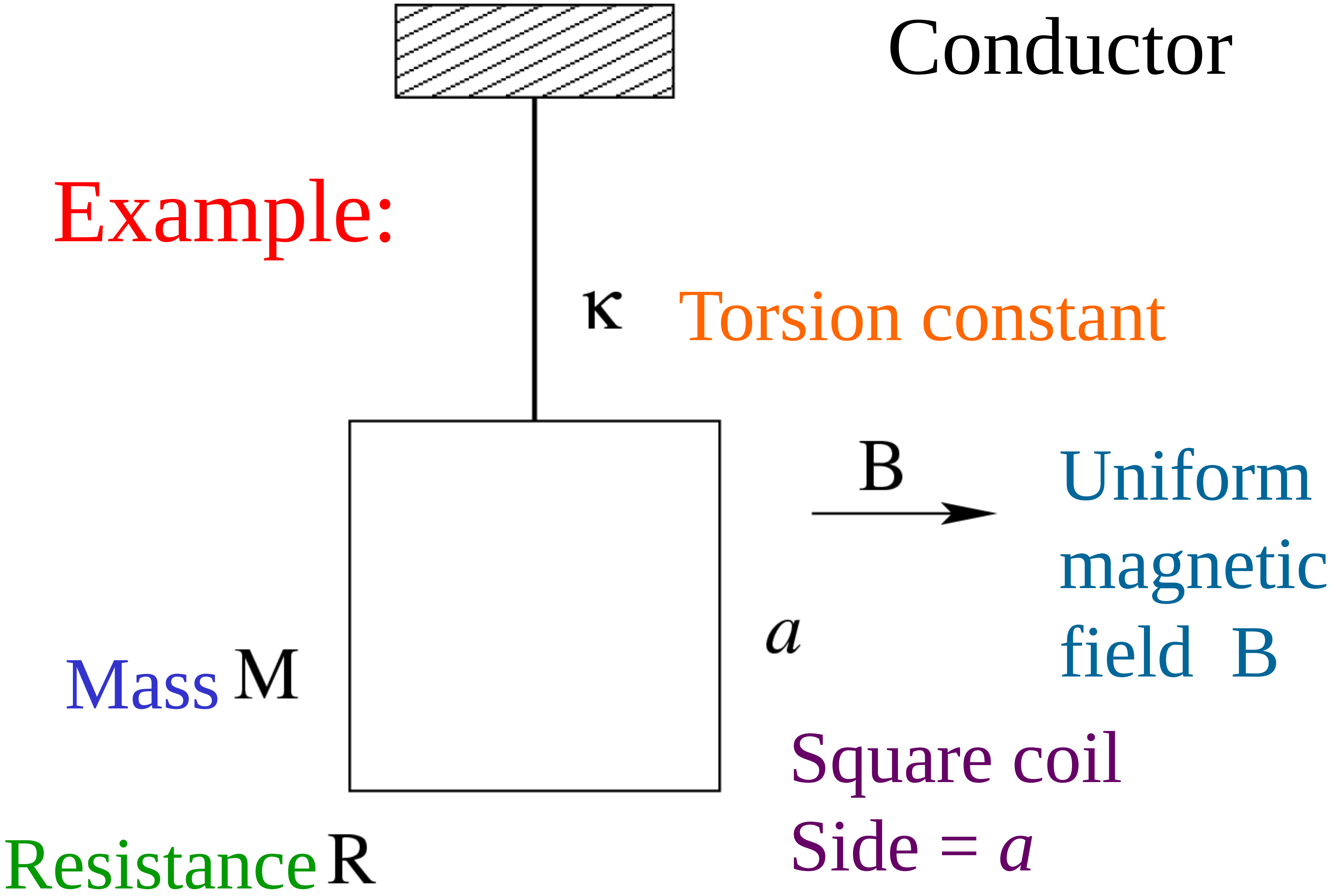
Example: LCR in series

Find charge on the capacitor at time t.



$$q(t) = q_{\pm}\exp(-Rt/2L \pm \sqrt{R^2/4L^2 - 1/LC}\; t)$$

Example:

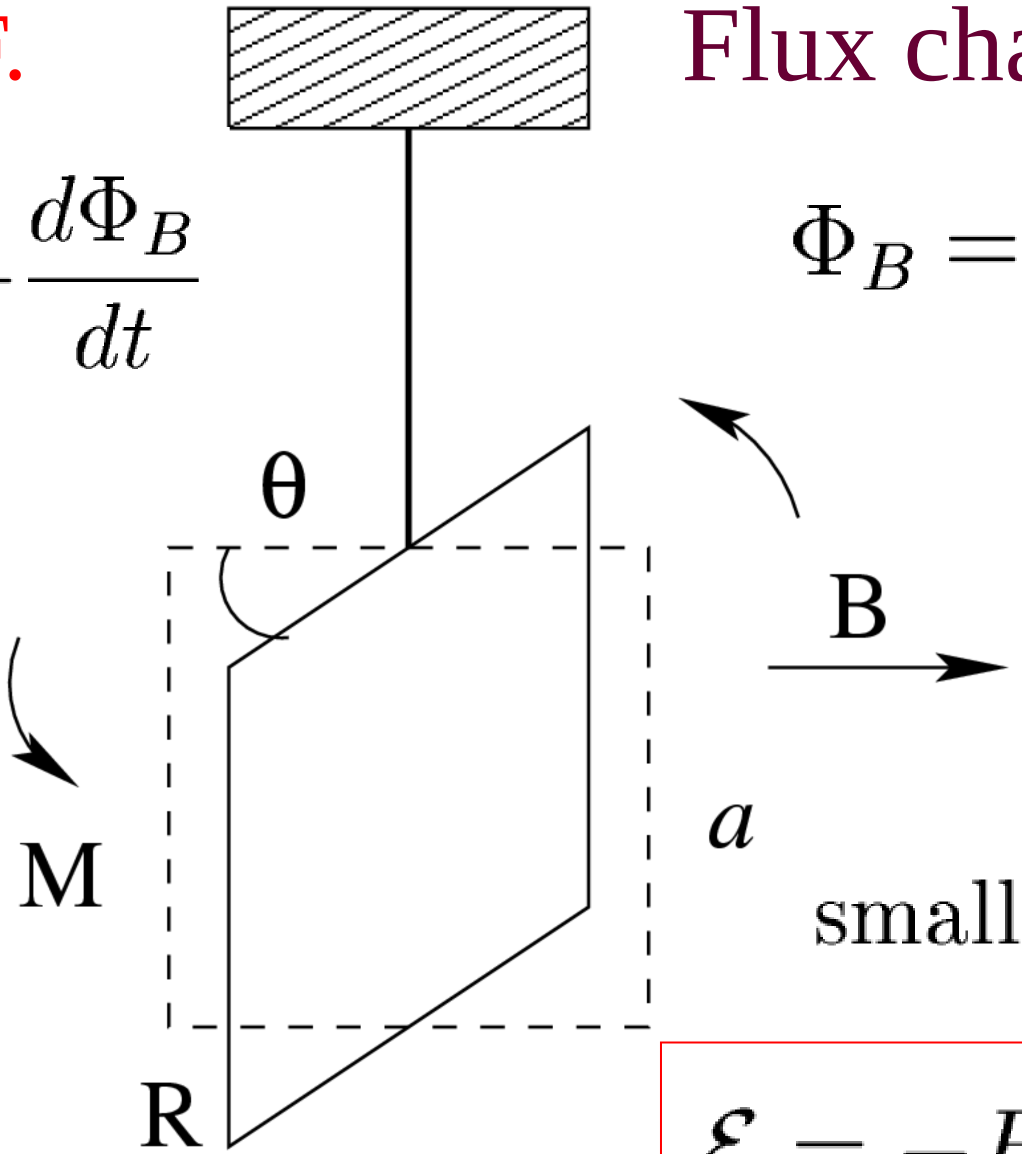

Conductor
κ
Torsion constant
B
Uniform magnetic field B
Mass M
a
Square coil
Side = a
Resistance R

E.M.F.

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

Flux change:

$$\Phi_B = Ba^2 \sin\theta$$



small θ, $\sin\theta \approx \theta$

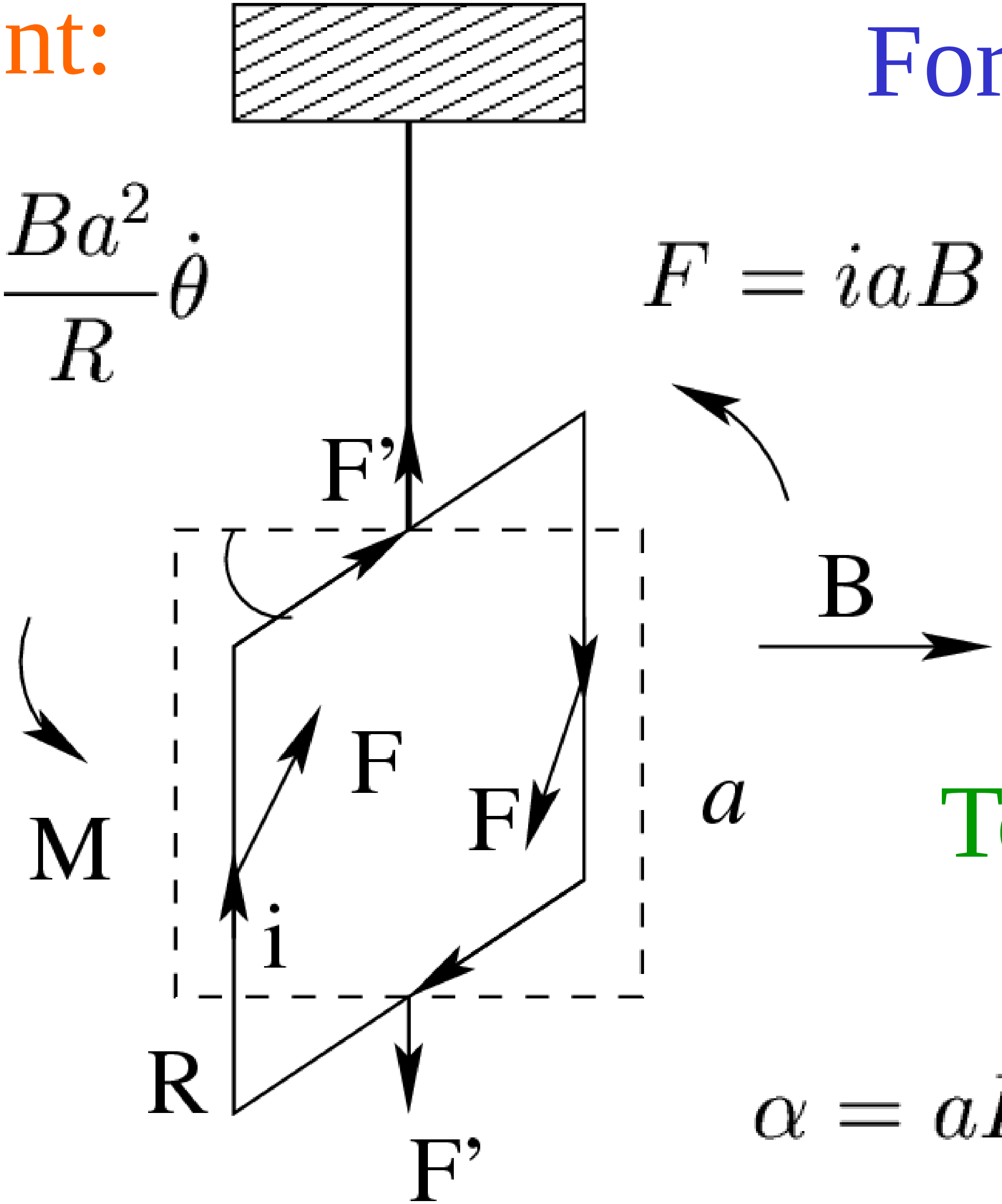
$$\mathcal{E} = -Ba^2\dot{\theta} = iR$$

Current:

$$i = -\frac{Ba^2}{R}\dot{\theta}$$

Force:

$$F = iaB = -\frac{B^2a^3}{R}\dot{\theta}$$



Torque:

$$\alpha = aF = -\frac{B^2a^4}{R}\dot{\theta}$$

$$I\ddot{\theta} = -\frac{B^2a^4}{R}\dot{\theta} - \kappa\theta$$

$$\ddot{\theta} + \frac{B^2a^4}{RI}\dot{\theta} + \frac{\kappa}{I}\theta = 0$$

$$2\beta = \frac{B^2a^4}{RI}$$

Relaxation time: $\tau = 1/\beta = \frac{2RI}{B^2 a^4}$

Moment of inertia: $I = Ma^2/6$

$$\tau = \frac{RM}{3B^2 a^2}$$