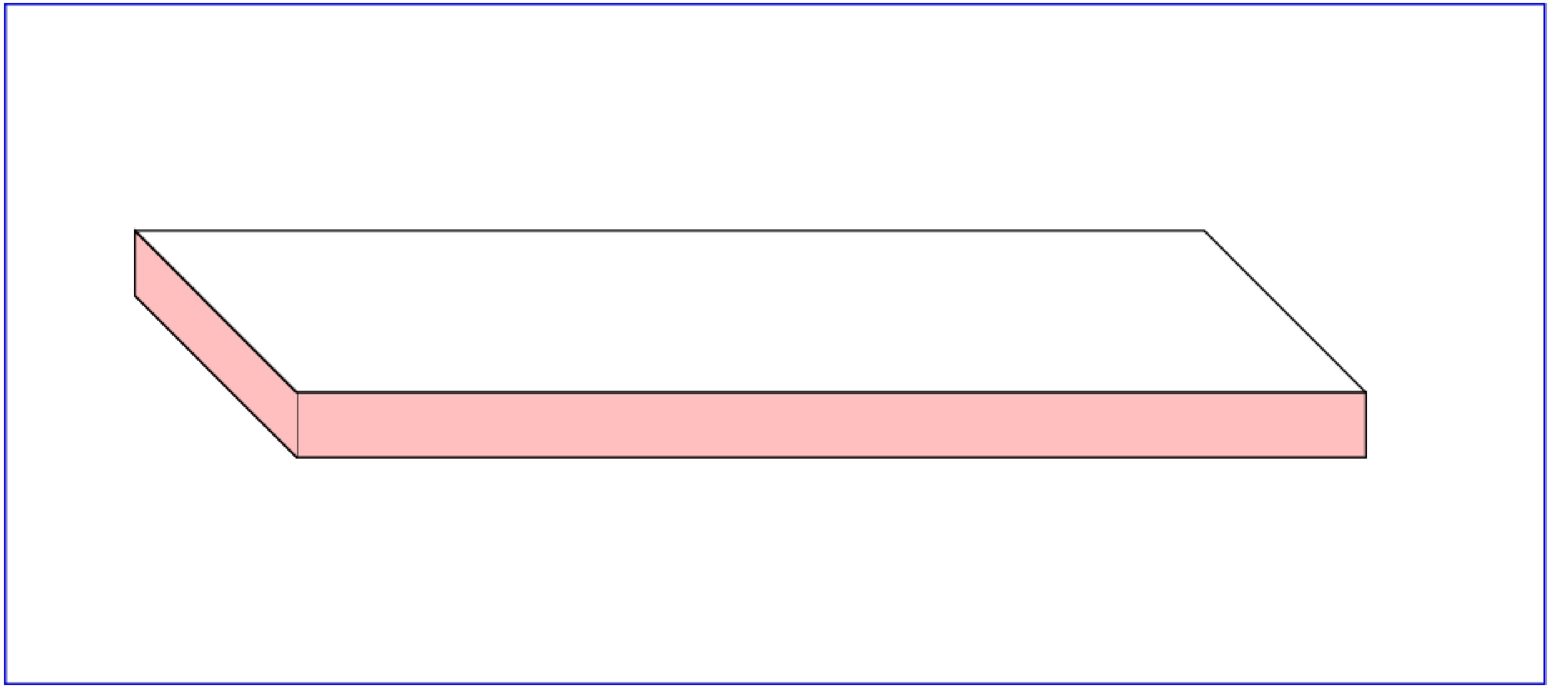
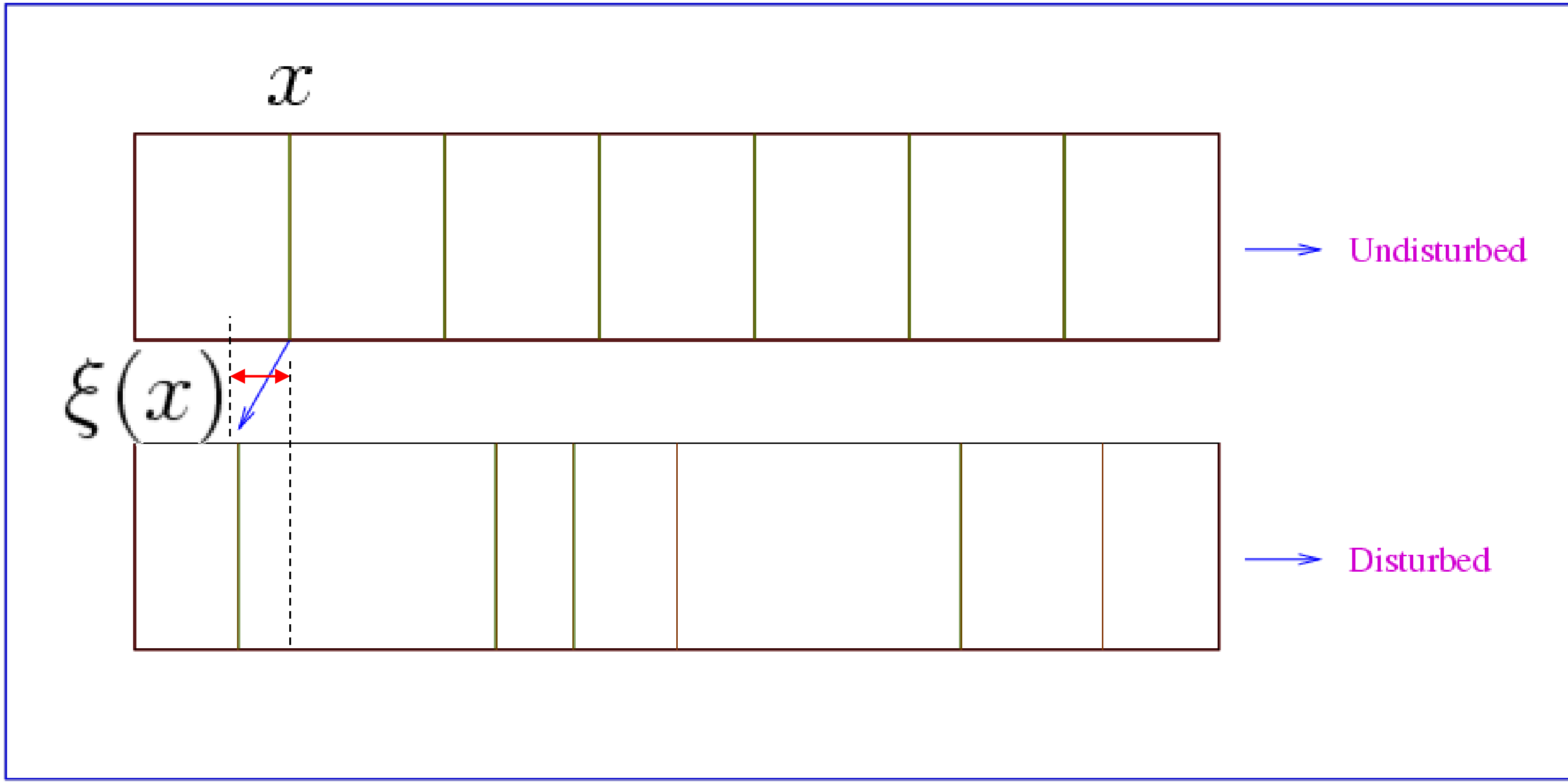
Waves

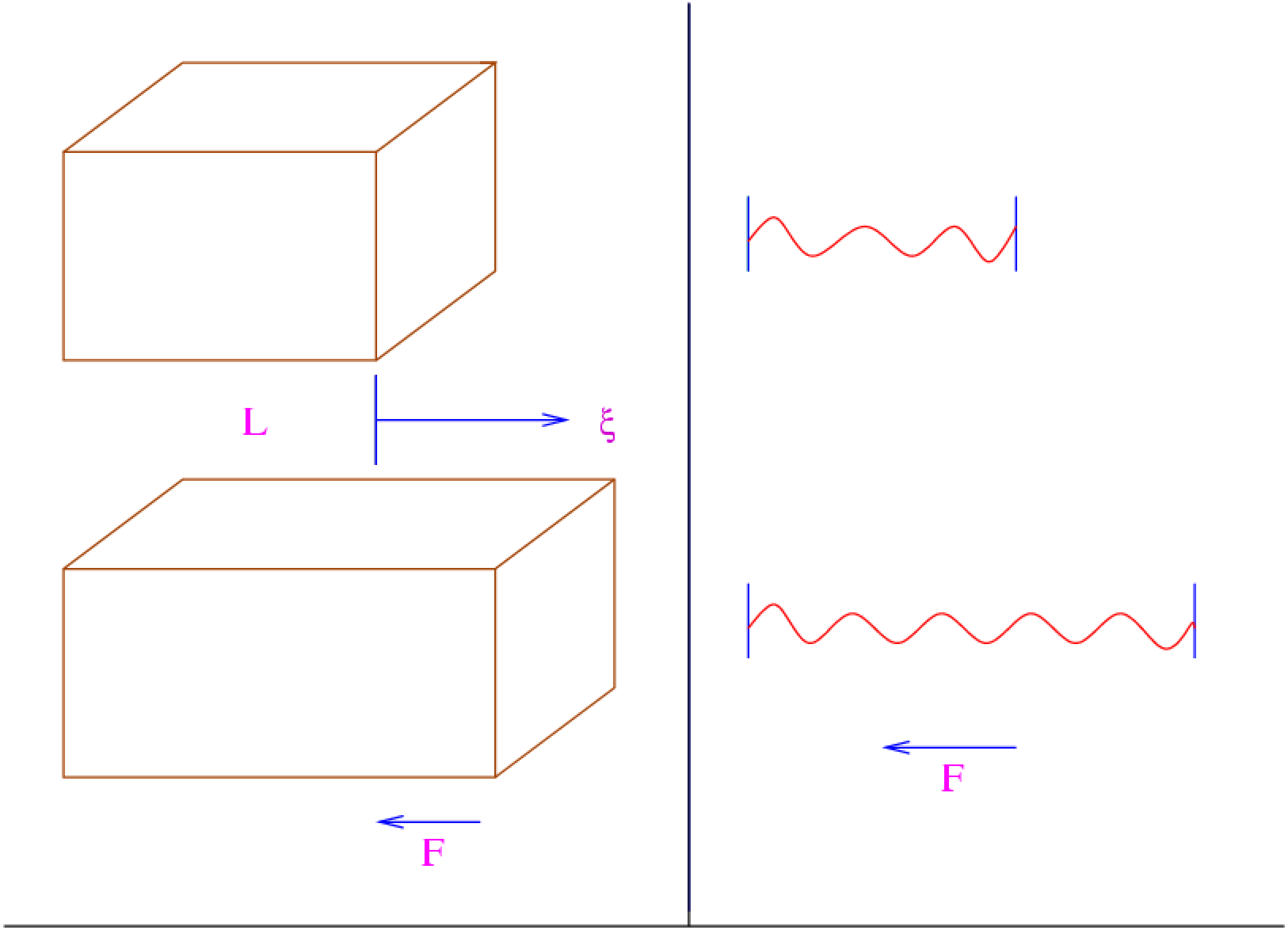
Longitudinal waves in an elastic medium

Shown below is a long rod of cross sectional area A.





$\xi(x)$ denotes the horizontal displacement of a point on the rod originally at x.


L
ξ
F
F

What happens to an elastic solid when it is compressed or stretched?

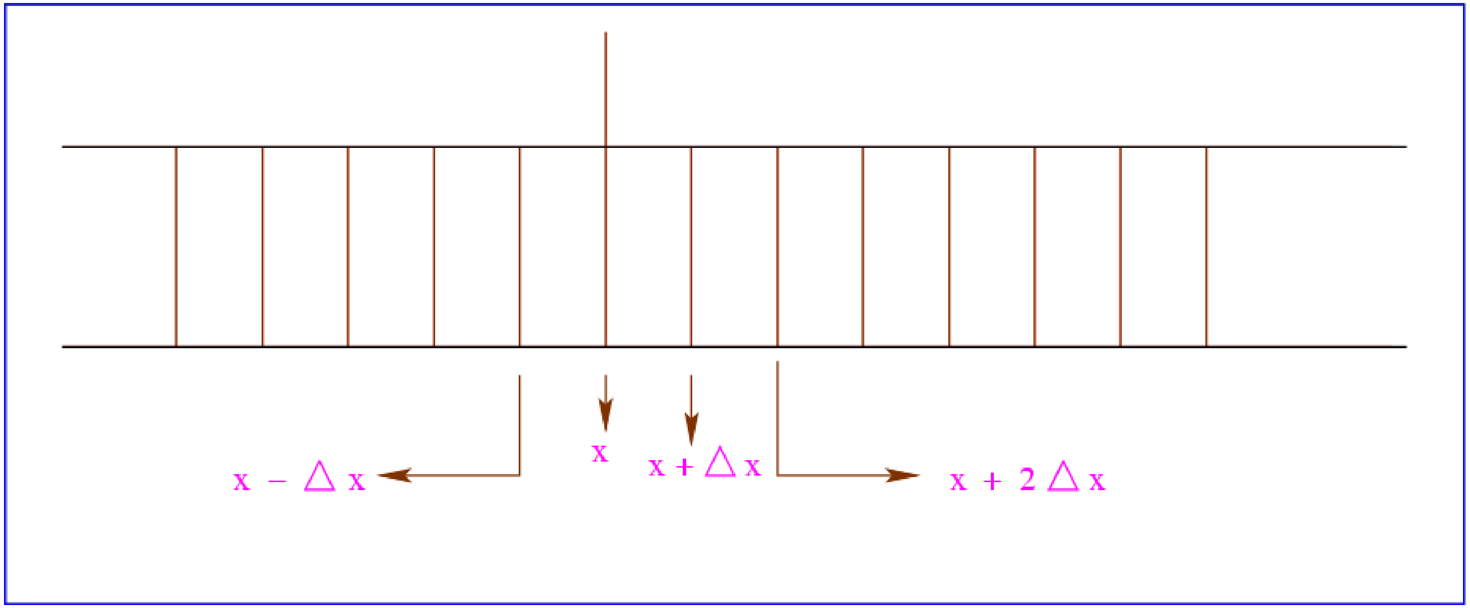
$$\text{Stress} = F/A$$

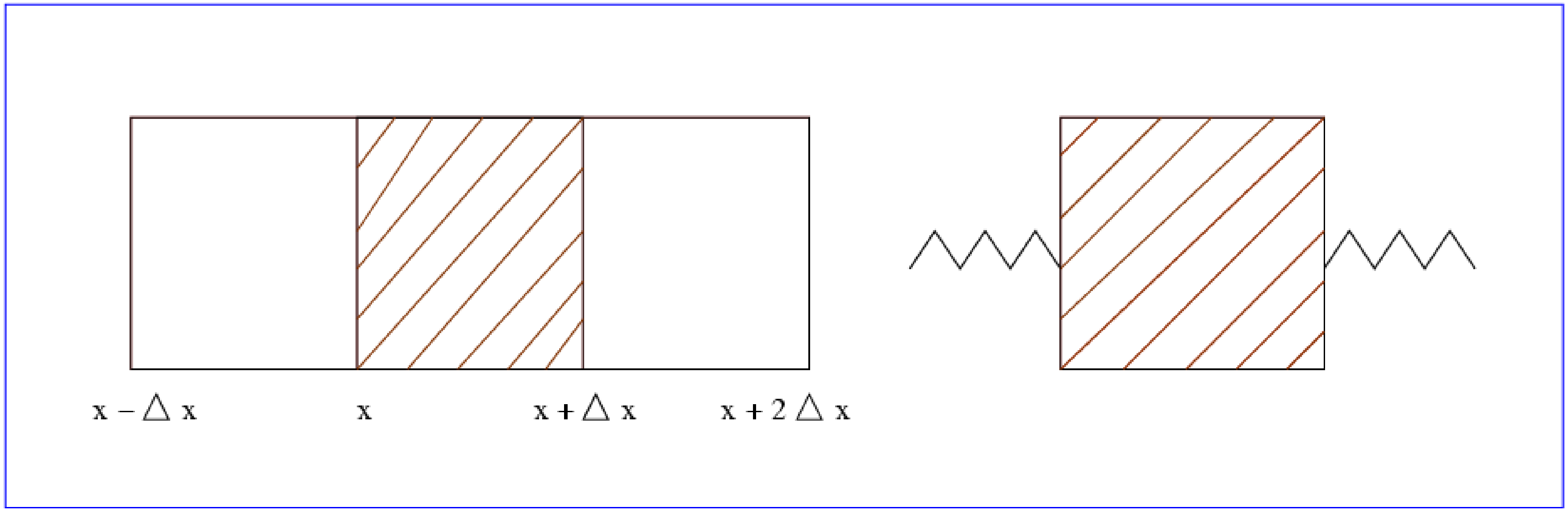
$$\text{Strain} = \frac{\xi}{L}$$

$$Y = \frac{Stress}{Strain} \quad \text{(Youngs modulus)}$$

$$F = \left(\frac{YA}{L}\right)\xi$$

$$F = kx \rightarrow \text{Spring}$$


x − △ x
x
x + △ x
x + 2 △ x


x − △ x
x
x + △ x
x + 2 △ x

Equation of motion

Force equation for slab *x*

$$\Delta x \;\; \varrho A \frac{\partial^2 \xi(x,t)}{\partial t^2} = F$$

compression of the slab on the right is

$$\xi(x+\Delta x) - \xi(x+2\Delta x)$$

Stress = F/A = Y . Strain

Force from the right on the slab

$$-\left(\frac{YA}{\Delta x}\right)[\xi(x+\Delta x,t)-\xi(x+2\Delta x,t)]$$

$$\approx \quad YA\frac{\partial\xi}{\partial x}(x+\Delta x,t)$$

Similarly the force from the left is

$$-\left(\frac{YA}{\Delta x}\right)[\xi(x,t)-\xi(x-\Delta x,t)]$$

$$=-YA\frac{\partial}{\partial x}\xi(x,t)$$

Total Force

$$=\quad YA\left[\frac{\partial}{\partial x}\xi(x+\Delta x,t)-\frac{\partial}{\partial x}\xi(x,t)\right]$$

$$= \quad YA \, \frac{\partial}{\partial x} \left[\xi(x + \Delta x, t) - \xi(x, t) \right]$$

$$= \quad YA\Delta x \frac{\partial}{\partial x} \left[\frac{1}{\Delta x} (\xi(x + \Delta x, t) - \xi(x, t)) \right]$$

$$= YA\Delta x \frac{\partial^2 \xi}{\partial x^2}$$

Force equation

$$\varrho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = Y A \Delta x \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial x^2} - \left(\frac{\varrho}{Y}\right) \frac{\partial^2 \xi}{\partial t^2} = 0$$

Wave equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$

Speed of the wave

$$c = \sqrt{\frac{Y}{\varrho}}$$

Solution of the wave equation

Change of variables

$$w_1 = x + ct \qquad w_2 = x - ct$$

$$x = \frac{w_1 + w_2}{2} \qquad t = \frac{w_1 - w_2}{2c}$$

$$
\begin{aligned}
\frac{\partial \xi}{\partial x} &= \frac{\partial w_1}{\partial x}\frac{\partial \xi}{\partial w_1} + \frac{\partial w_2}{\partial x}\frac{\partial \xi}{\partial w_2} \\
&= \frac{\partial \xi}{\partial w_1} + \frac{\partial \xi}{\partial w_2}
\end{aligned}
$$

$$
\frac{1}{c}\,\frac{\partial \xi}{\partial t} = \frac{\partial \xi}{\partial w_1} - \frac{\partial \xi}{\partial w_2}
$$

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2}\,\frac{\partial \xi}{\partial t^2}$$

$$= \left(\frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}\right)^2 \xi - \left(\frac{\partial}{\partial w_1} - \frac{\partial}{\partial w_2}\right)^2 \xi = 0$$

$$\Rightarrow \quad \frac{\partial^2}{\partial w_1 \partial w_2}\xi(w_1, w_2) = 0$$

Possible solutions:-

1. $\xi(w_1, w_2) = \text{Constant}$ (not of interest)

2. $\xi(w_1, w_2) = f_1(w_1)$ (function of w_1 alone.)

3. $\xi(w_1, w_2) = f_2(w_2)$ (function of w_2 alone.)

Superposition of 2 and 3

$$\xi(w_1, w_2) = c_1 f(w_1) + c_2 f(w_2)$$

Interpretation of solution 2

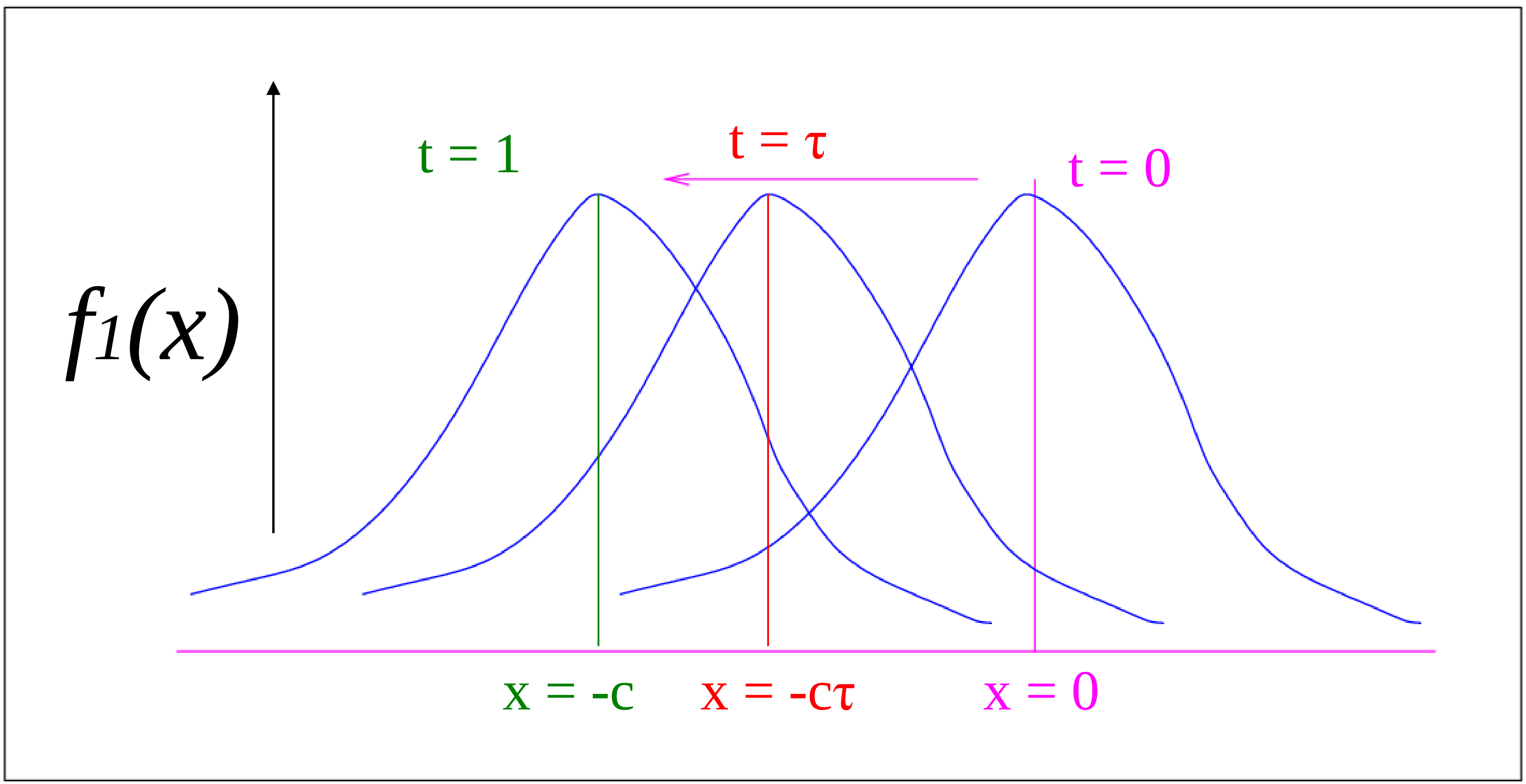
arbitrary function of $w_1 = x + ct.$

$$\xi(x,t) = f(x + ct)$$

At t = 0, $\xi(x,0) = f_1(x)$

At t = 1, $\xi(x,1) = f_1(x + c)$

Wave moving left



Similarly solution 3 is a wave moving right

$$\xi(x,t) = f_2(x - ct)$$