Coupled Oscillators

Consider two identical simple harmonic oscillators of mass m and spring constant k as shown in Figure 1 (a.). The two oscillators are independent with

$$x_0(t) = a_0 \cos(\omega t + \phi_0) \tag{1}$$

and

$$x_1(t) = a_1 \cos(\omega t + \phi_1) \tag{2}$$

where they both oscillate with the same frequency $\omega = \sqrt{\frac{k}{m}}$. The amplitudes a_0, a_1 and the phases ϕ_0, ϕ_1 of the two oscillators are in no way interdependent. The question which we take up for discussion here is what happens if the two masses are coupled by a third spring as shown in Figure 1 (b.).

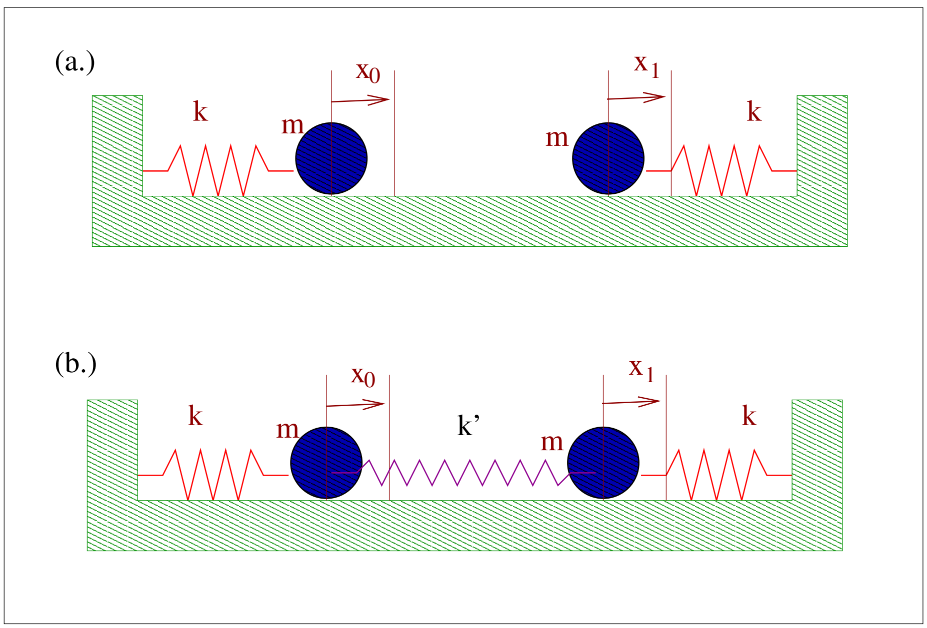

Figure 1: This shows two identical spring-mass systems. In (a.) the two oscillators are independent whereas in (b.) they are coupled through an extra spring.

The motion of the two oscillators is now coupled through the third spring of spring constant $k^{'}$. It is clear that the oscillation of one oscillator affects the second. The phases and amplitudes of the two oscillators are no longer independent and the frequency of oscillation is also modified. We proceed to calculate these effects below.

The equations governing the coupled oscillators are

$$m\frac{d^2 x_0}{dt^2} = -kx_0 - k^{'}(x_0 - x_1) \tag{3}$$

and

$$m\frac{d^2\,x_1}{dt^2} = -kx_1 - k^{'}(x_1 - x_0) \tag{4}$$

The technique to solve such *coupled differential equations* is to identify linear combinations of x_0 and x_1 for which the equations become decoupled. In this case it is very easy to identify such variables

$$q_0 = \frac{x_0 + x_1}{2} \text{ and } q_1 = \frac{x_0 - x_1}{2}\,. \tag{5}$$

These are referred to as as the *normal modes* of the system and the equations governing them are

$$m\frac{d^2\,q_0}{dt^2} = -kq_0 \tag{6}$$

and

$$m\frac{d^2\,q_1}{dt^2} = -(k + 2k^{'})q_1\,. \tag{7}$$

In this case the normal modes lend themselves to a simple physical interpretation where.

The normal mode q_0 represents the center of mass. The center of mass behaves as if it were a particle of mass $2m$ attached to two springs (Figure 2) and its oscillation frequency is the same as that of the individual uncoupled oscillators $\omega_0 = \sqrt{\frac{2k}{2m}} = \omega$.

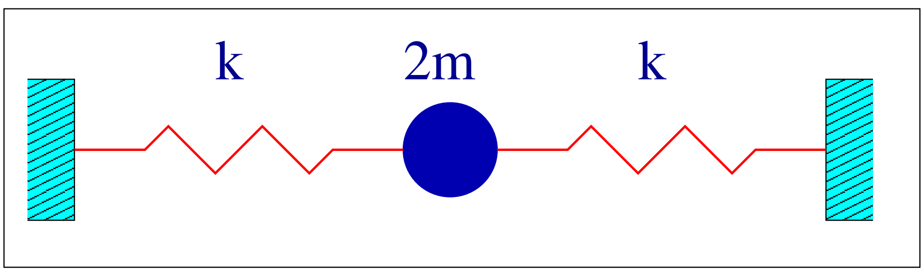

Figure 2: This shows the spring mass equivalent of the normal mode q_0 which corresponds to the center of mass.

The normal mode q_1 represents the relative motion of he two masses which leaves the center of mass unchanged. This can be thought of as the motion of two two particles of mass m connected to a spring of spring constant $\tilde{k} = (k + 2k^{'})/2$ as shown in Figure 3. The oscillation frequency of this normal mode $\omega_1 = \sqrt{\frac{k+2k^{'}}{m}}$ is always higher than that of the individual uncoupled oscillators (or the center of mass).

Equivalently, we may interpret q_0 as a mode of oscillation where the two masses oscillate with exactly the same phase, and q_0 as a mode where they have a phase difference of π (Figure 4). Recollect that the phases of the two masses are independent when the two masses are not coupled. Introducing a coupling causes the phases to be interdependent.

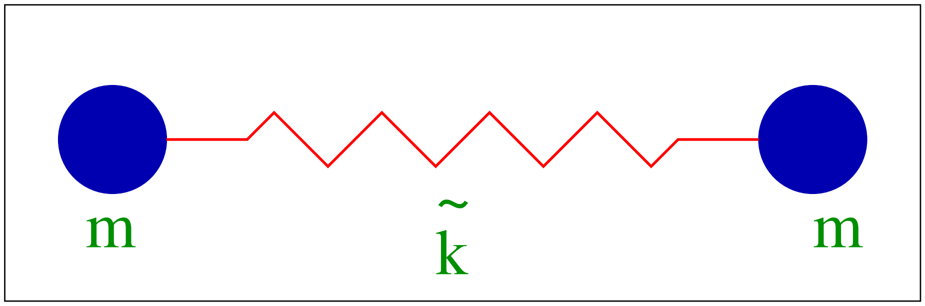

Figure 3: This shows the spring mass equivalent of the normal mode q_1 which corresponds to two particles connected through a spring.

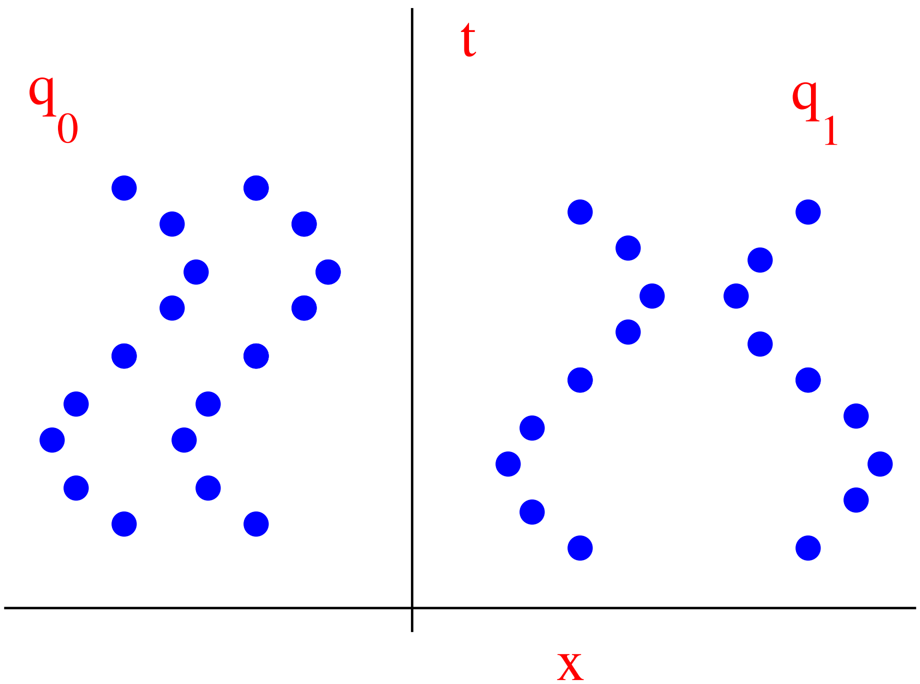

Figure 4: This shows the motion corresponding to the two normal modes q_0 and q_1 respectively.

Solution

For the normal modes we have

$$q_0(t) = A_0 \; e^{i \; \omega_o t} \tag{8}$$

$$q_1(t) = A_1 \; e^{i \; \omega_1 t} \tag{9}$$

where it should be borne in mind that A_0 and A_1 are complex numbers with both amplitude and phase ie. $A_0 = |A_0| \, e^{i\psi_0}$ etc.

$$x_0(t) = |A_0| \, e^{i(\omega_0 t + \psi_0)} + |A_1| \, e^{i(\omega_1 t + \psi_1)} \tag{10}$$

better to write it as

$$x_0(t) = A_0\ e^{i\ \omega_0 t} + A_1\ e^{i\ \omega_1 t} \tag{11}$$

$$x_1(t) = A_0\ e^{i\ \omega_0 t} - A_1 e^{i\ \omega_1 t} \tag{12}$$

Examples.

(1) The two particles are initially at rest. The particle x_0 is given a small displacement and then left to oscillate.

$$x_1\ (0) = 0 \Rightarrow A_0 = A_1 \tag{13}$$

$$x_0(t) = A_0 \left[e^{i\ \omega_0 t} + e^{i\ \omega_1 t}\right] \tag{14}$$

$$\dot{x}_0(0) = 0 \Rightarrow\ \mathrm{Re}\left\{A_0\left[i\ \omega_0 + i\ \omega_1\right]\right\} = 0 \tag{15}$$

$\Rightarrow A_0$ is real, which we denote by $\frac{a}{2}$ the initial displacement of the first particle.

$$x_0(t) = \frac{a_0}{2}\left[\cos\ \omega_0 t + \cos\ \omega_1 t\right] \tag{16}$$

$$x_1(t) = \frac{a_0}{2}\left[\cos\ \omega_0 t - \cos\ \omega_1 t\right] \tag{17}$$

$$x_0(t) = a_0\ \cos\left[\left(\frac{\omega_1 - \omega_0}{2}\right)t\right]\cos\left[\left(\frac{\omega_0 + \omega_1}{2}\right)t\right] \tag{18}$$

$$x_1(t) = a_0\ \sin\left[\frac{(\omega_1 - \omega_0)t}{2}\right]\sin\left[\left(\frac{\omega_0 + \omega_1}{2}\right)t\right] \tag{19}$$

(A) **Stiff coupling** $\quad K' \gg K$

This is like connecting the two masses with a nearly rigid rod.

$$\omega_1 \gg \omega_0$$

The two oscillation frequencies are far apart. If we observe the motion on a small time scale $\sim \frac{1}{\omega_1}$, we will not see the slow oscillation.

If we observe on a long time scale $\sim \frac{1}{\omega_0}$ the fast oscillation cancels out and we see only the slow oscillation.

Weak coupling $\quad K' \ll K$

$$\omega_1 = \sqrt{\frac{K}{m}\left(1 + \frac{K'}{K}\right)} \approx \omega_0 + 2\Delta\omega \text{ where } \frac{\Delta\omega}{\omega_0} = \frac{K'}{4K} \ll 1 \tag{20}$$

$$x_0(t) = a\ \cos\ \Delta\omega t\ \cos\ \omega_0 t \tag{21}$$

$$x_1(t) = a\ \sin\ \Delta\omega t\ \sin\ \omega_0 t \tag{22}$$

A. Both particles oscillate with ω_0, the amplitude undergoes a slow modulation at angular frequency $\Delta\omega$.

B. The oscillations are slowly transferred from the particle which receives the initial displacement to the particle originally at rest, and then back again. The roles of driver and driven are interchanged

Appendix: Coupled oscillations are also realised by connecting a spring between two simple pendula.

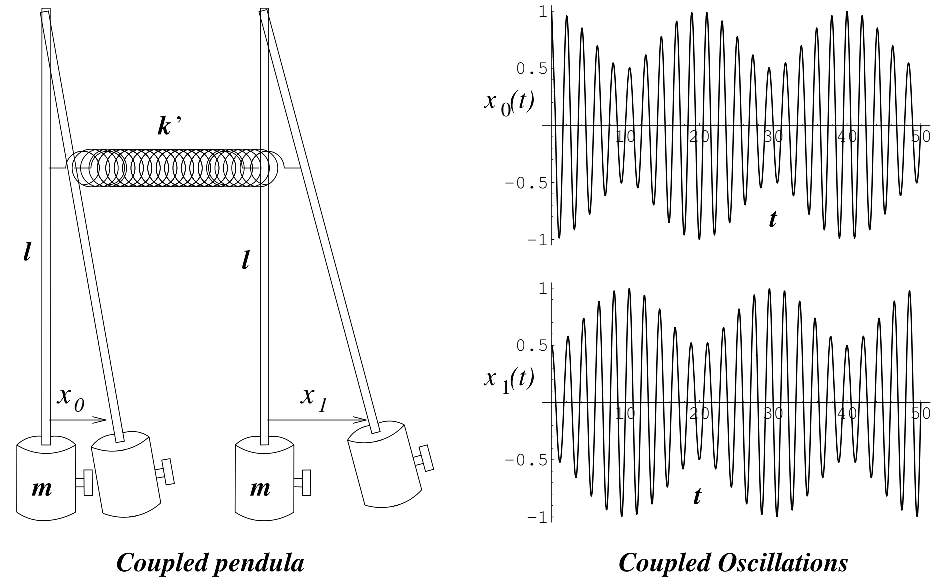

Figure 5: Coupled pendula

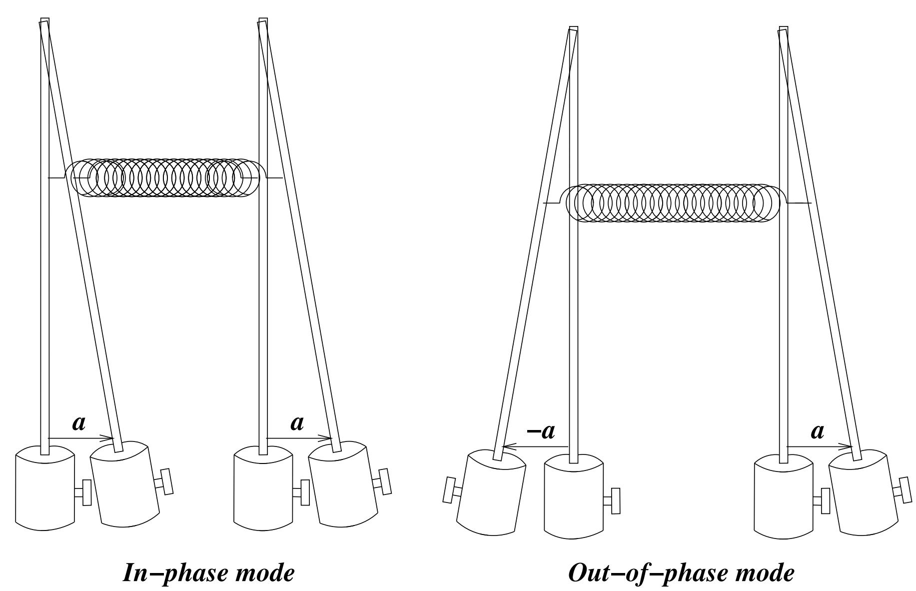

Figure 6: In-phase and out-of-phase mode

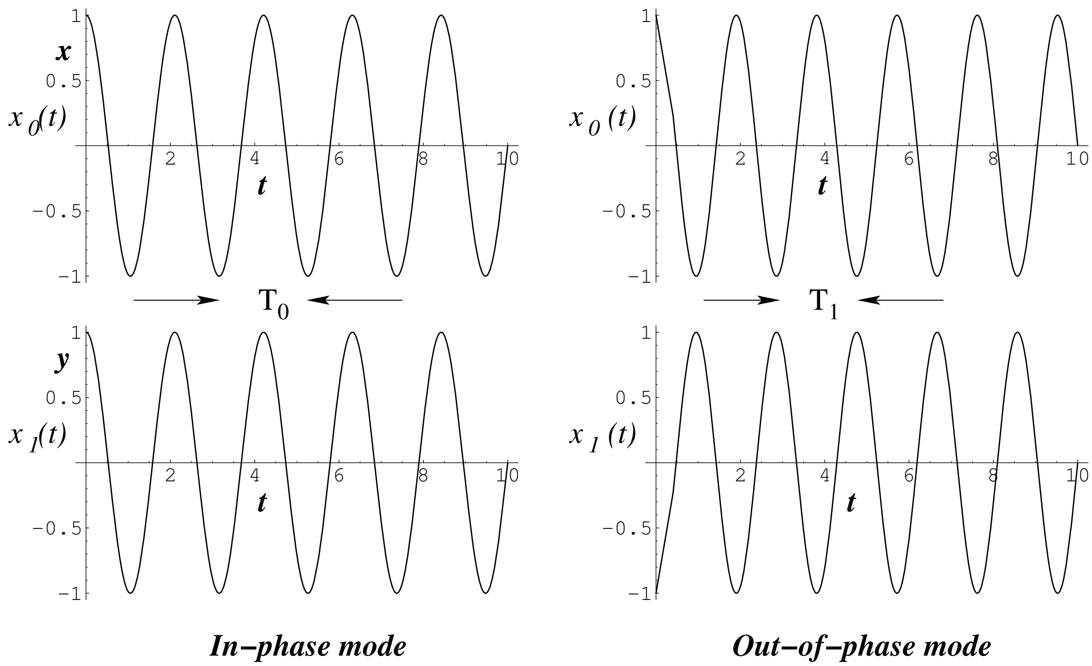

Figure 7: In-phase and out-of-phase evolution

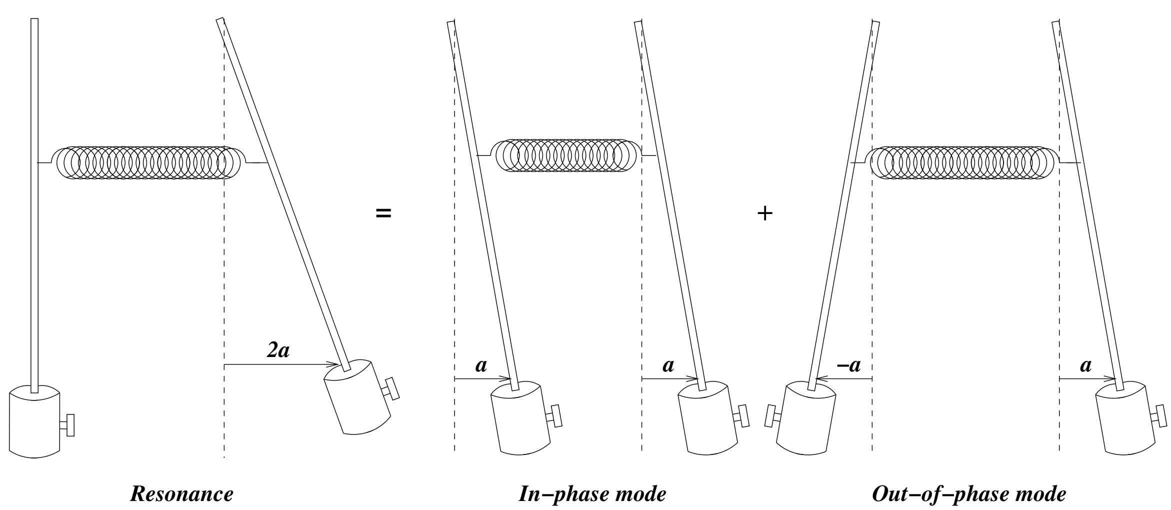

Figure 8: Resonance

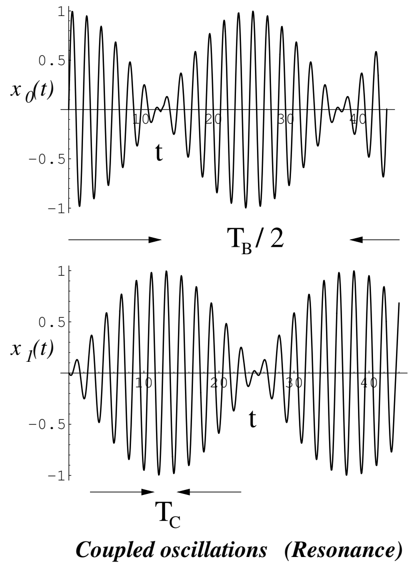

Figure 9: Resonance

Problem 1: Construct a coupled system with two masses and two springs.
Solution: There can be several solutions one is shown below.
*Problem 2:*Construct a coupled system with one mass and one spring.
Solution: Shown in the figure. In this case there is coupling between the rotation motion and the up-down translational motion of the system. One has to adjust mass cleverly for this system to observe the rotational energy changing into the translational and vice-versa.

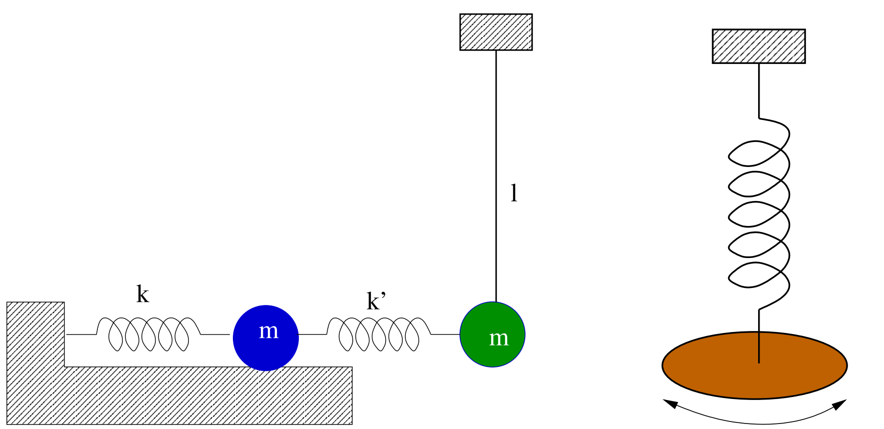

Figure 10: Coupled systems