Forced (Driven) simple harmonic motion

Undamped forced oscillations: As we have seen in the last chapter that due to the resistance oscillations eventually die down. To maintain the oscillations one needs a driving force. First we shall study the forced oscillations without the damping term. So the basic equation of motion in this case is

$$m\ddot{x} + kx = F(t) \tag{1}$$
$$\ddot{x} + \omega_0^2 x = F(t)/m. \tag{2}$$

The general solution of the equation (2) is given as sum of two parts. First part is known as *particular solution*, say $P(t)$, which satisfies the equation (2). The second part, known as *complementary function*, say $C(t)$, is the solution of the equation (2) with right hand side set to zero (i.e. solution of ordinary SHM). Writing in terms of equations we have,

$$\frac{d^2P(t)}{dt^2} + \omega_0^2 P(t) = F(t)/m, \tag{3}$$
$$\frac{d^2C(t)}{dt^2} + \omega_0^2 C(t) = 0, \tag{4}$$

Adding equations (3) and (4) we have,

$$\frac{d^2(P(t) + C(t))}{dt^2} + \omega_0^2(P(t) + C(t)) = F(t)/m. \tag{5}$$

So $x(t) = P(t) + C(t)$ gives the general solution of the equation (2) with two arbitrary constants coming form the complementary function and determined by initial conditions.

We shall now assume a sinusoidal time dependence ($F(t) = F_0 \sin \omega t$) of forcing. The angular frequency *omega* appearing in the driving force is called the *driving frequency*. Why we would like to study the sinusoidal forcing will become clear as we proceed. So the equation we are interested in solving, is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \sin \omega t = f_0 \sin \omega t, \tag{6}$$

where, $F_0/m = f_0$. Since we already know the complementary function we look for a particular solution. We try a solution of the type $P(t) = A \sin \omega t$. Substituting this $P(t)$ in the equation (6), we get

$$-A\omega^2 \sin \omega t + A\omega_0^2 \sin \omega t = f_0 \sin \omega t. \tag{7}$$

Above equation (7) finds the amplitude,

$$A = \frac{f_0}{(\omega_0^2 - \omega^2)}. \tag{8}$$

The general solution can now be written as by adding the complementary function

$$x(t) = \frac{f_0}{\omega_0^2 - \omega^2} \sin \omega t + B \cos \omega_0 t + C \sin \omega_0 t. \tag{9}$$

We find that as the driving frequency ω approaches the natural frequency ω_0 from below the phenomenon of resonance occurs and the amplitude A tends to ∞. Once it crosses ω_0 the amplitude tends to $-\infty$. Now since we always consider amplitude as a positive quantity we define the amplitude as $|A|$ and compensate the $-$ve sign of amplitude for $\omega > \omega_0$ including a phase in the argument of the sine function of particular solution. For this we are left with two choices in hand $\sin(\omega t - \pi)$ and $\sin(\omega t + \pi)$. We cannot *a priori* decide whether the oscillations lead or lag the driving force. We take the hint from damped forced oscillations (which is to be done in the next section) and settle for $\sin(\omega t - \pi)$ for $\omega > \omega_0$. So in this case there is an abrupt change of $-\pi$ radians in the phase. So the first term of(9) is written as.

$$x(t) = \frac{f_0}{\omega^2 - \omega_0^2} \sin(\omega t - \pi), \qquad (\omega > \omega_0) \tag{10}$$

The amplitude and the phase are plotted against the frequency below.

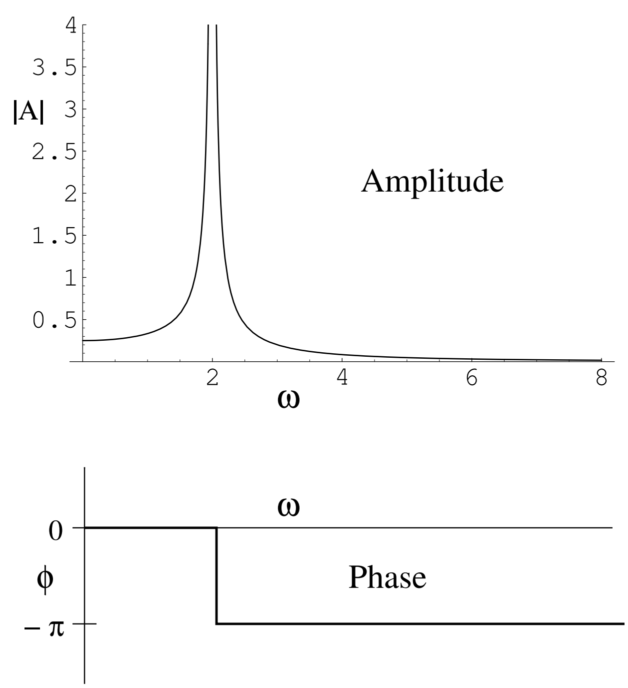


Figure 1: Amplitude and phase of undamped driven oscillator as a function of driving frequency ω.

We fix B and C using some initial conditions. Let us choose $x(t = 0) = \dot{x}(t = 0) = 0$. The condition $x(0) = 0$ fixes $B = 0$ and $\dot{x}(0)$ finds $C = -\frac{f_0 \omega}{\omega_0(\omega_0^2 - \omega^2)} = -A\omega/\omega_0$.

Hence the solution (9) becomes,

$$x(t) = A(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t) \tag{11}$$

Following is a sample plot of behaviour away from resonance, We would now like

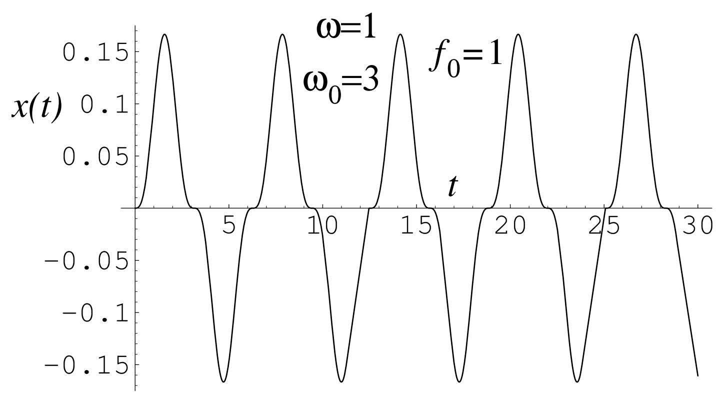


Figure 2: Undamped forced oscillations away from resonance

to investigate the behaviour of general solution (9) near resonance. Let us take $\omega = \omega_0 - \Delta\omega$,

$$\begin{aligned} x(t) &= A(\sin \omega_0 t \cos \Delta\omega t - \cos \omega_0 t \sin \Delta\omega t - \frac{\omega}{\omega_0} \sin \omega_0 t) && (12) \\ x(t) &= A\left(\frac{(\omega_0 - \omega)}{\omega_0} \sin \omega_0 t - \Delta\omega t \cos \omega_0 t\right) && (13) \end{aligned}$$

where in equation (13) we have used $\cos \Delta\omega t \approx 1$ and $\sin \Delta\omega t \approx \Delta\omega t$. Substituting the value of A from (8) we get,

$$\begin{aligned} x(t) &= \frac{f_0}{\omega_0(\omega_0 + \omega)}(\sin \omega_0 t - \omega_0 t \cos \omega_0 t) && (14) \\ &\approx \frac{f_0}{2\omega_0^2}(\sin \omega_0 t - \omega_0 t \cos \omega_0 t) && (15) \end{aligned}$$

Damped forced oscillations: Now we include the damping term in the l.h.s of equation (2) and the equation becomes,

$$\begin{aligned} m\ddot{x} + 2r\dot{x} + kx &= F(t) && (16) \\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= F(t)/m. && (17) \end{aligned}$$

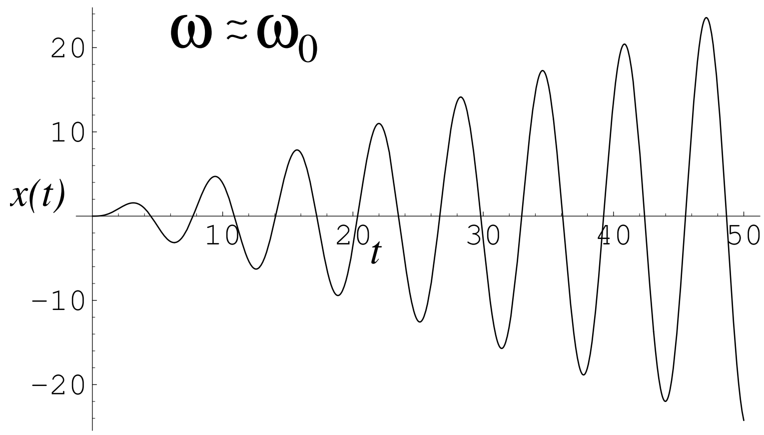


Figure 3: Undamped forced oscillations, behaviour near resonance

Again we shall again restrict ourselves to sinusoidal forcing. We shall start with $F(t) = F_0 \cos \omega t$. The equation becomes

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t = f_0 \cos\omega t, \tag{18}$$

The above equation can be solved by an elegant method, which we shall describe it now. We write a companion equation of the equation (18) by changing the forcing on the r.h.s. by $F_0 \sin \omega t$,

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = \frac{F_0}{m}\sin\omega t = f_0 \sin\omega t, \tag{19}$$

By multiplying the equation (19) by i and adding to the equation (18), we have,

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}\exp(i\omega t) = f_0 \exp(i\omega t), \tag{20}$$

where we have defined $z = x + iy$. Now if we solve the equation (20) and separate the real and imaginary parts we would have the solutions for equations (18) and (19). Now as described earlier the general solution of (20) will have two parts viz. the *particular solution* and the *complementary function* and the latter we have already solved in the last chapter so we find the particular solution. We also notice that if we wait for longer times we would have the *steady state solution* which is basically the particular solution. The *complementary function* gives the so called *transient* which die down if one waits a little longer. For a steady state solution of the equation (20) we try the form

$$z(t) = z_0 \exp(i\omega t), \tag{21}$$

where z_0 is a complex constant. Substituting this in (20), we have,

$$z_0(-\omega^2 + 2i\beta\omega + \omega_0^2) = f_0, \tag{22}$$

giving

$$z_0 = \frac{f_0}{(\omega_0^2 - \omega^2 + 2i\beta\omega)}. \tag{23}$$

Now writing $z_0 = |z_0| \exp(i\phi)$, we get

$$|z_0| = \sqrt{z_0 z_0^*} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{24}$$

and

$$\phi = \tan^{-1}\left(\frac{-2\beta\omega}{\omega_0^2 - \omega^2}\right). \tag{25}$$

Hence the particular solution is given by

$$z(t) = |z_0| \exp(i(\omega t + \phi)) = |z_0|(\cos(\omega t + \phi) + i\sin(\omega t + \phi)) \tag{26}$$

where $|z_0|$ and ϕ are given by (24) and (25) respectively. From (26) we can read off the solutions of (18) and (19) as,

$$x(t) = |z_0| \cos(\omega t + \phi) \tag{27}$$

and

$$y(t) = |z_0| \sin(\omega t + \phi) \tag{28}$$

respectively. The plots of amplitudes and phase as a function of driving frequency is given in the figure (4). Now we can justify the choice of phase change of $-\pi$ at $\omega > \omega_0$ for the case of undamped forced oscillator considered in the previous section.

Case 1: **Low frequencies:** $\omega << \omega_0$**:** In low frequencies the response is Hooke's law like. It is stiffness dominated.

$$|z_0|_{st} \rightarrow \frac{f_0}{\omega_0^2} = \frac{F_0}{k}$$

$$\phi \rightarrow 0$$

$$x(t) \approx \frac{F_0}{k} \cos\omega t$$

Case 2: **High frequencies:** $\omega >> \omega_0$**:** High frequency response is mass dominated. The oscillator is totally out of phase with the driver.

$$|z_0| \rightarrow \frac{f_0}{\omega^2}$$

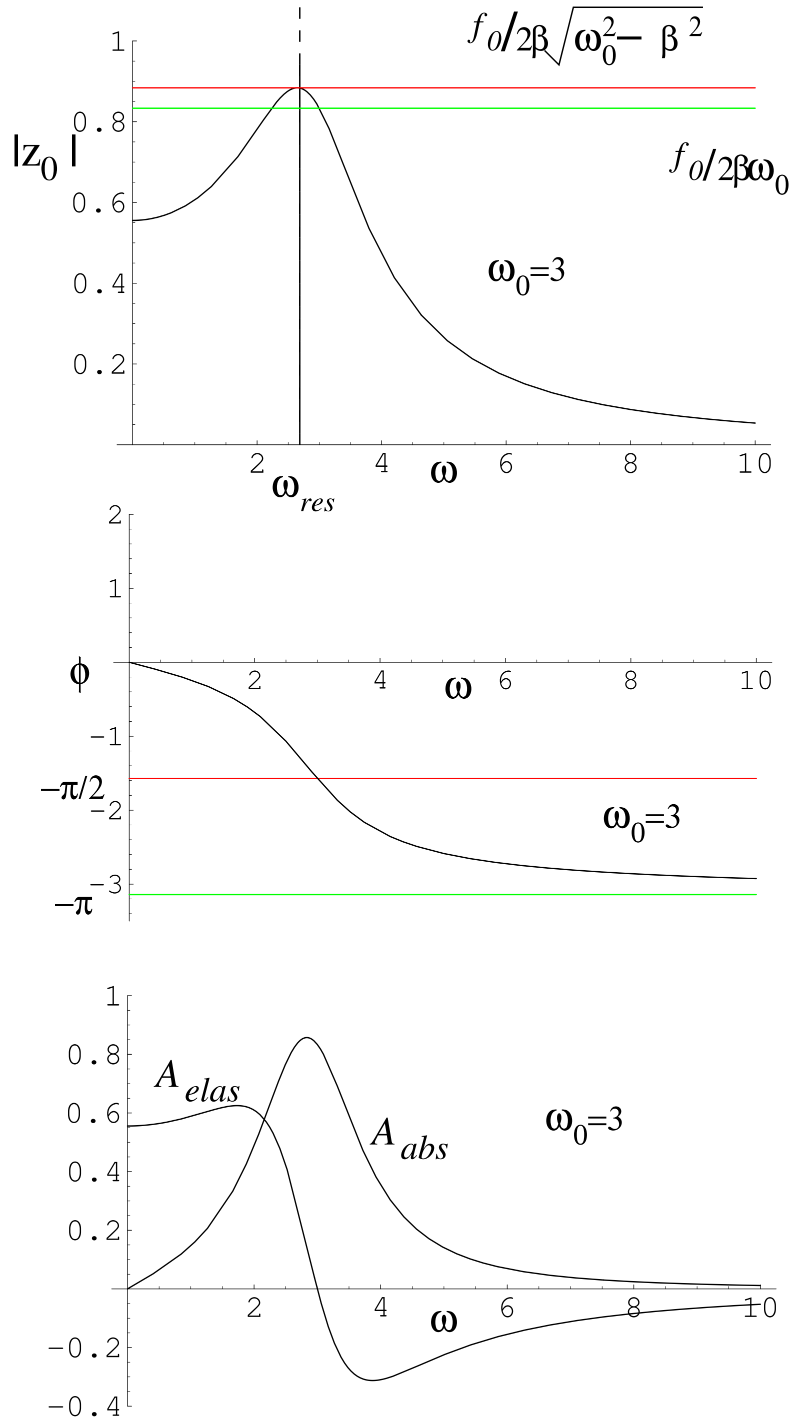


Figure 4: Amplitudes and phase of damped forced oscillator

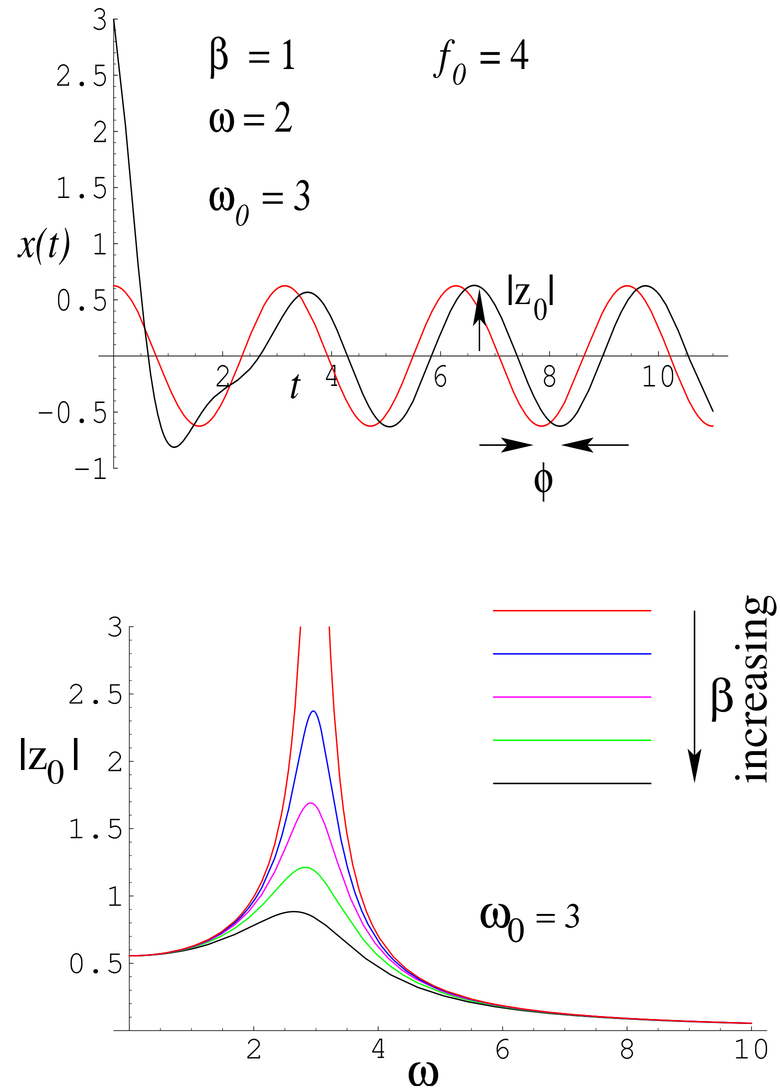


Figure 5: Forced oscillation amplitudes and phase for different resistances

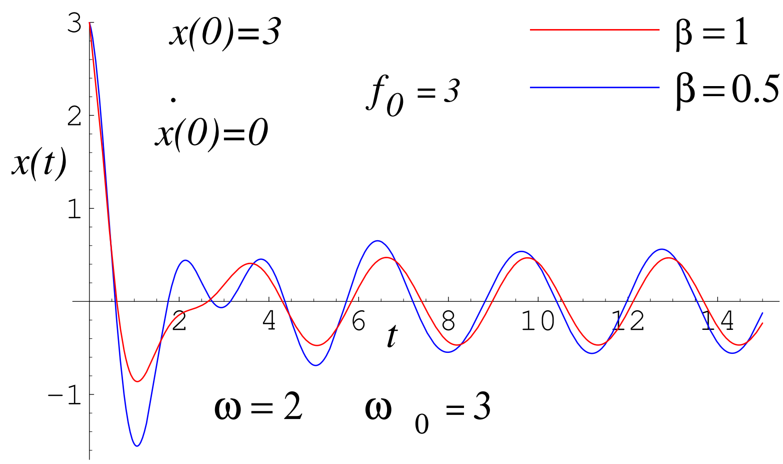


Figure 6: Forced oscillations with diffrent resistances

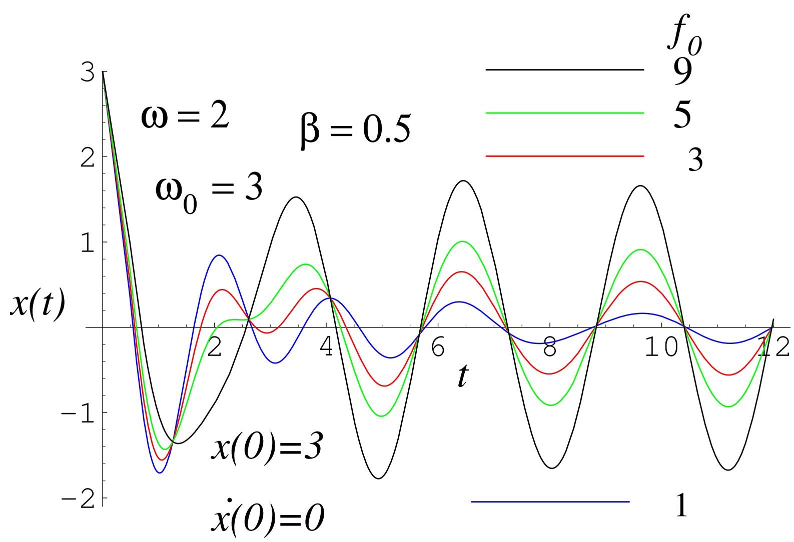


Figure 7: Forced oscillations with diffrent driving amplitudes

$$\phi \to -\pi$$

$$x(t) \approx -\frac{F_0}{m\omega^2}\cos\omega t$$

Case 3: **Intermediate frequencies: $\omega \approx \omega_0$:** In the intermediate frequency regime the response is resistance dominated.

For $\omega = \omega_0$

$$|z_0| = \frac{f_0}{2\beta\omega_0}$$

$$\phi \to -\frac{\pi}{2}$$

$$x(t) = \frac{F_0}{2m\beta\omega_0}\sin\omega_0 t$$

Resonance: The amplitude resonance happens for

$$\omega_{res}^2 = \omega_0^2 - 2\beta^2.$$

If β is very small compared to ω_0 resonant frequency ω_{res} approaches natural frequency of the system.

$$|z_0|_{res} = \frac{f_0}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

The solution $|z_0|\cos(\omega t + \phi)$ can be written as,

$$x(t) = A_{abs}\sin\omega t + A_{elas}\cos\omega t \tag{29}$$

where, $A_{elas} = |z_0|\cos\phi$ and $A_{abs} = -|z_0|\sin\phi$. The amplitude A_{abs} is called the absorptive amplitude because the average power is non zero from the first term and is zero for the second. Let us now calculate the instantaneous power absorbed by the damped forced oscillator (18). It is given by,

$$P(t) = F_0\cos(\omega t)\dot{x}(t) = -|z_0|F_0\omega\cos(\omega t)\sin(\omega t + \phi) \tag{30}$$

We find the power averaged over one cycle,

$$< P >= F_0\omega A_{abs} < \cos^2\omega t > -F_0\omega A_{elas} < \sin\omega t\cos\omega t > \tag{31}$$

where $<\quad>$ denotes average value. We know that

$$< \cos^2\omega t >\equiv \frac{1}{T}\int_0^T dt\ \cos^2\omega t = \frac{1}{2T}\int_0^T dt(1 + \cos 2\omega t) = \frac{1}{2}, \tag{32}$$

where T is the time period. Similarly, it is easy to show that $< \sin\omega t\cos\omega t >= 0$.

$$< P >= F_0\omega A_{abs}/2 = -\frac{F_0\omega|z_0|\sin\phi}{2}$$

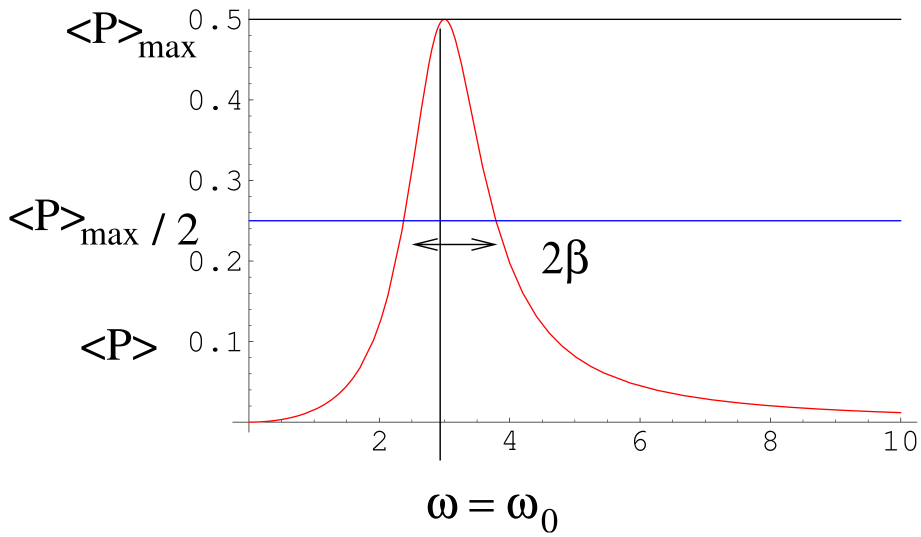


Figure 8: Average input power curve

$$< P >= r f_0^2 \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{33}$$

Power peak is found for $\omega^2 = \omega_0^2$

$$< P >_{max} = \frac{r f_0^2}{4\beta^2} = \frac{F_0^2}{4r}.$$

At half maximum,

$$\omega = \sqrt{\omega^2 + \beta^2} \pm \beta$$

Averaged stored energy Full width at half maximum for the average power curve is found to be equal to 2β.

Problem: Show that the average power loss due to the resistance dissipation is equal to the average input power calculated above in (33).

$$< E >= \frac{m}{4}(\omega^2 + \omega_0^2)|z_0|^2$$

$$< E >= \frac{m f_0^2}{4} \frac{(\omega^2 + \omega_0^2)}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}$$

Problem: Find the angular frequency for peak energy and value of peak energy.

Resonant quality factor is defined many ways:

$$Q_0 = |z_0|_{res}/|z_0|_{st}$$

$$Q_0 = \frac{\omega_0}{2\beta}$$

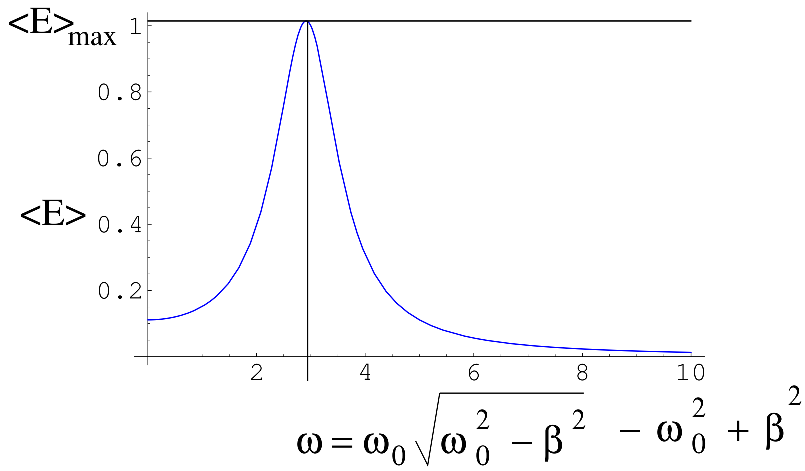


Figure 9: Average energy curve

Example 1: **Galvanometer:** A galvanometer is connected through a switch with a direct-current source of constant EMF. At time t=0, the switch is closed. After a sufficientlylong time the galvanometer deflection reaches its final value $\theta_{max.}$ What is its motion between the initial position of rest, $\theta = 0$, $\dot{\theta} = 0$, and the final position$\theta = \theta_{max}$? Take damping torque proportional to angular velocity. Distinguish and explain graphically underdamped,critically damped and overdamped cases. *Solution:* We solve the forced oscillator equation with constant forcing (i.e. driving frequency =0) and given intial conditions and plot the various evolutions.

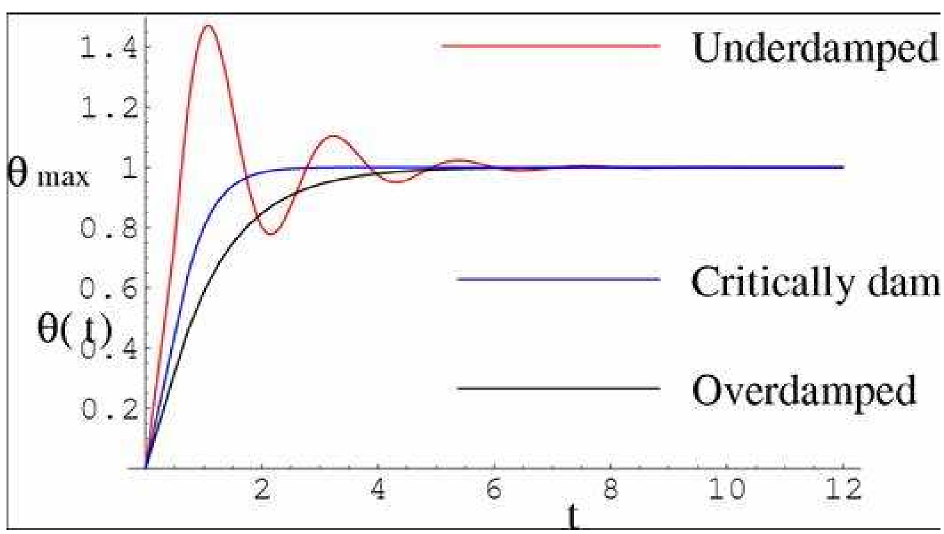
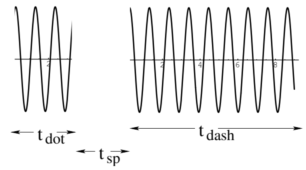


Figure 10: Galvanmeter deflection

Example 2: **Telegraph:** A radio receiver receives radio telegraph signals in Morse code in the form of sinusoidal wave packets. a) The inductance of the circuit is 100



μH, the capacitance is 250 pF and the resistance is 0.2 ohm. Find the interval between the impulses tsp needed to prevent two adjacent signals from merging.

b) Assuming the duration of the 'dot' signal to be $t_{dot} = 1.5t_{sp}$ and that of the 'dash' $t_{dash} = 4.5t_{sp}$ find the maximum amount of information that can be transmitted per unit time.

Solution: a) t_{sp} = twice the relaxation time is a safe margin because by this time the amplitude will decay by nearly a factor of 10 and hence two separate signals will be distinguished properly.

$$t_{sp} = 2\tau = 2/\beta = 4L/R = 2 \times 10^{-3} \text{ sec} = 2 \text{ msec}.$$

b) Let maximum of N pulses can be sent per sec. Then on an average there shall be $N/2$ 'dots' and $N/2$ 'dashes', so

$$Nt_{sp} + \frac{N}{2}t_{dot} + \frac{N}{2}t_{dash} = 4Nt_{sp} = 1$$

giveing $N = 125$.

Appendix: Complex plane and phasors

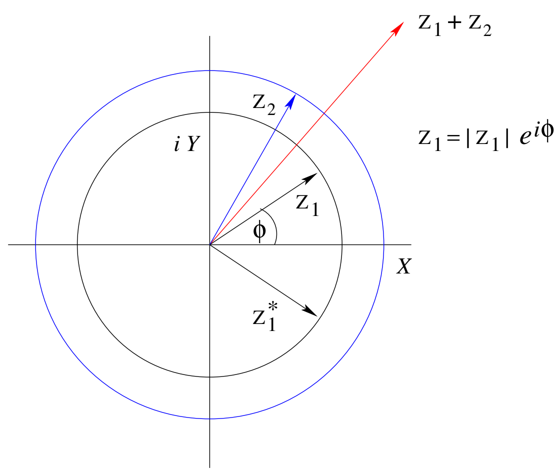


Figure 11: Phasors in complex plane