Damped simple harmonic motion

In reality there is always a resistance in any system. We neglected this when we were dealing with SHM and that was an idealistic case easy to solve. Here we will study SHM with damping. We shall restrict ourselves to a certain type of damping viz the cases where the resistance is proportional to the velocity. There are two reasons for studying this, firstly many of the realistic systems do have a viscous damping proportional to the velocity and secondly in this case the equation of motion remains linear and hence again has simple solutions. In its most general form the equation will be written as

$$m\ddot{x} + 2r\dot{x} + kx = 0 \tag{1}$$

$$\ddot{x} + \frac{2r}{m}\dot{x} + \frac{k}{m}x = 0 \tag{2}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{3}$$

where we have substituted $\beta = r/m$ and natural frequency $\omega_0 = \sqrt{k/m}$. In equation (3) second term in the left hand side is the damping term or the resistance which is proportional to the velocity $(\dot{x})$. Now depending upon the strength of the resistance there can be various types of solution possible viz. overdamped, critically damped or underdamped.

We try solutions of the form $x = A\exp(\gamma t)$. Substituting the above form of x in the equation (3) we get,

$$A\gamma^2 \exp(\gamma t) + 2\beta A\gamma \exp(\gamma t) + \omega_0^2 A \exp(\gamma t) = 0, \tag{4}$$

giving the following equation relating various parameters of the system.

$$\gamma^2 + 2\beta\gamma + \omega_0^2 = 0. \tag{5}$$

Equation (5) gives two roots for γ as,

$$\gamma = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{6}$$

Case I: Overdamped ($\beta^2 > \omega_0^2$) In this case the discriminant is real we have two decaying solutions. A general solution can be written as a linear combination of these two solutions,

$$x(t) = \exp(-\beta t)[A_1 \exp(\sqrt{\beta^2 - \omega_0^2}\ t) + A_2 \exp(-\sqrt{\beta^2 - \omega_0^2}\ t)] \tag{7}$$

The initial conditions will decide the values of constants A_1 and A_2. There will not be any oscillations. The above solution can also be written in the following way,

$$x(t) = \exp(-\beta t)[A\cosh(\sqrt{\beta^2 - \omega_0^2}\ t) + B\sinh(\sqrt{\beta^2 - \omega_0^2}\ t)], \tag{8}$$

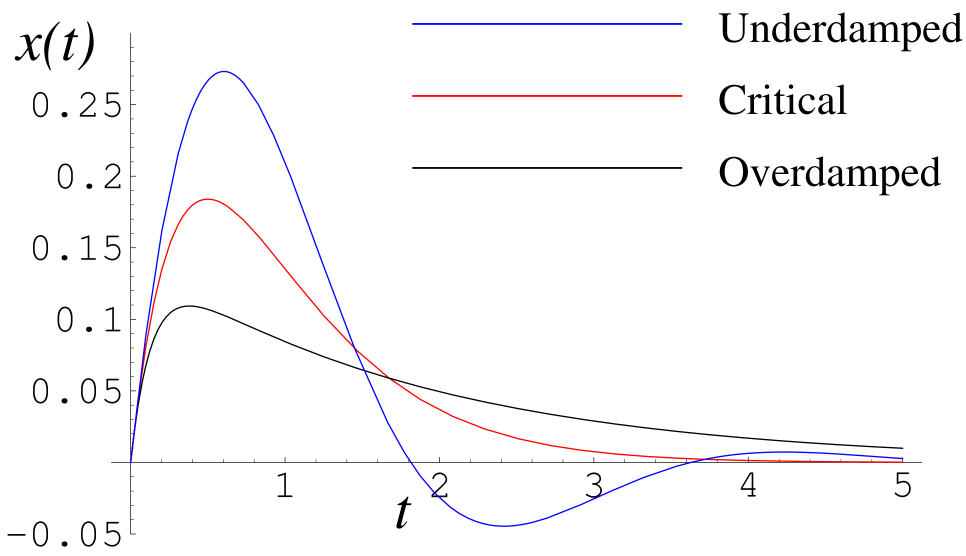

Figure 1: Underdamped, critically damped and overdamped cases

using hyperbolic functions.[1] We have,

$$A_1 = (A+B)/2, \qquad A_2 = (A-B)/2.$$

Initial conditions give,

$$x(0) = A_1 + A_2 = A.$$

Calculating the velocity and using again the initial conditions we have,

$$\begin{aligned} \dot{x}(t) &= A_1(-\beta + \sqrt{\beta^2 - \omega_0^2}) \exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\ t) && (9) \\ &+ A_2(-\beta - \sqrt{\beta^2 - \omega_0^2}) \exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\ t) && (10) \end{aligned}$$

$$\dot{x}(0) = -\beta(A_1 + A_2) + \sqrt{\beta^2 - \omega_0^2}\ (A_1 - A_2)$$

$$\frac{\dot{x}(0) + \beta x(0)}{\sqrt{\beta^2 - \omega_0^2}} = (A_1 - A_2) = B$$

$$A_1 = \frac{1}{2}\left[x(0) + \frac{\dot{x}(0) + \beta x(0)}{\sqrt{\beta^2 - \omega_0^2}}\right]$$

$$A_2 = \frac{1}{2}\left[x(0) - \frac{\dot{x}(0) + \beta x(0)}{\sqrt{\beta^2 - \omega_0^2}}\right]$$

[1] See appendix for hyperbolic functions

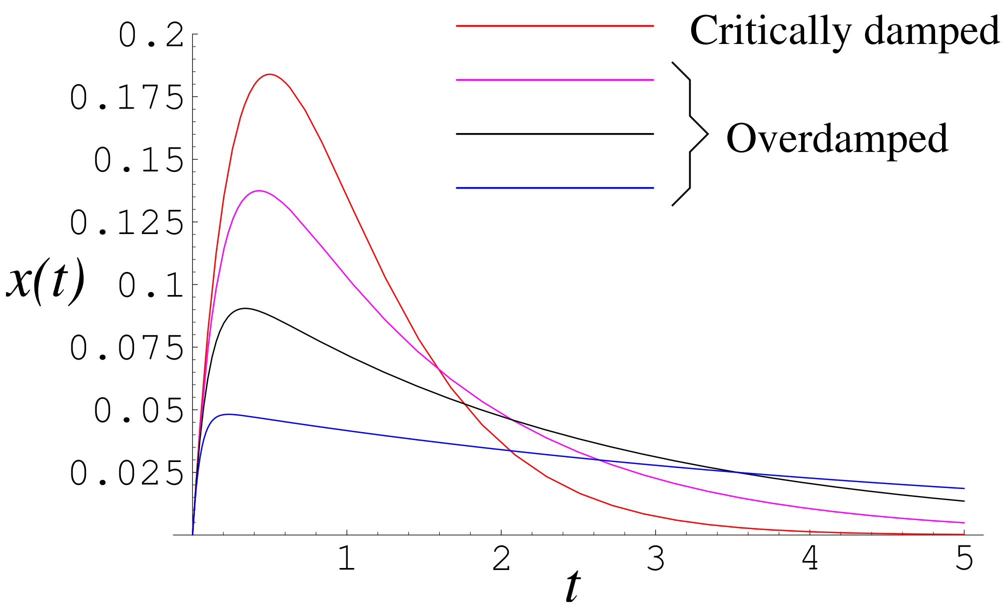

Figure 2: Overdamped cases for diffrent resistances

Case II: Critically damped ($\beta^2 = \omega_0^2$) In this case two solutions for γ converge to one. This solution is given by $A \exp(-\beta t)$. The other solution is found to be $B\ t \exp(-\beta t)$. Combining these two solution we have

$$x(t) = (A + Bt)\exp(-\beta t) \tag{11}$$

Again A and B are fixed by initial conditions. Like overdamped case in this case also we do not have oscillations. Once can show that in critically damped case it takes least time to reach the mean position from the maximum displacement for the system. In galvanometers nearly this condition is maintained so that the needle can come to the zero position as soon as possible avoiding the oscillations once current becomes zero.

Problem: Show the solution for critical damping as limiting cases of overdamped solution.

Case III: Underdamped ($\beta^2 < \omega_0^2$) Here discriminant is negative and hence we have two oscillating solutions.

$$x(t) = \exp(-\beta t)[A_1 \exp(-i\sqrt{\omega_0^2 - \beta^2}\ t) + A_2 \exp(i\sqrt{\omega_0^2 - \beta^2}\ t)] \tag{12}$$

We find $x(t)^*$, the complex conjugate of $x(t)$. The $*$ is a complex conjugate operation. We have then,

$$x(t)^* = \exp(-\beta t)[A_1^* \exp(i\sqrt{\omega_0^2 - \beta^2}\ t) + A_2^* \exp(-i\sqrt{\omega_0^2 - \beta^2}\ t)] \tag{13}$$

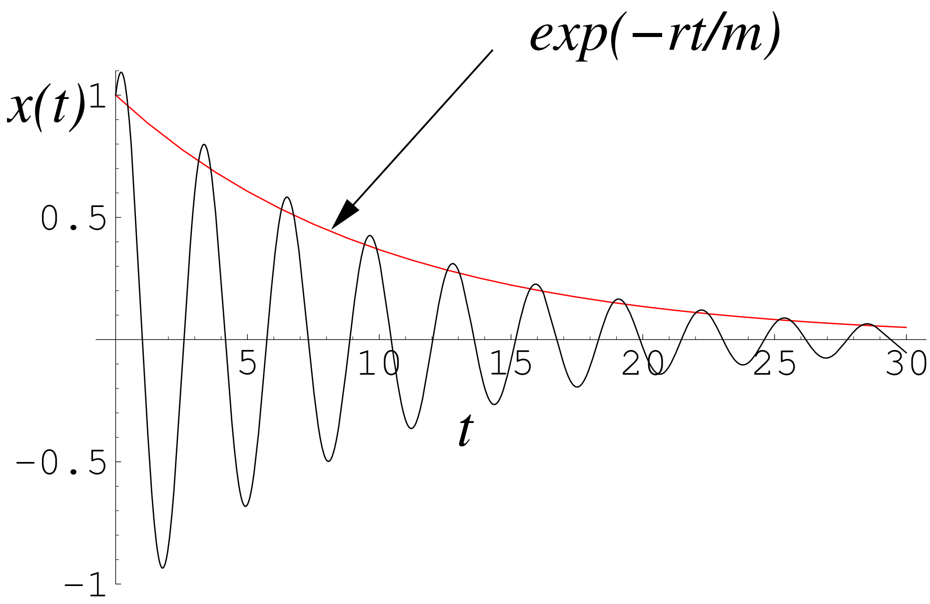

Figure 3: Underdamped oscillations

Since $x(t)$ is real for all time. We have $x(t) = x(t)^*$. This puts the following restriction on the amplitudes,

$$A_1^* = A_2, \qquad A_2^* = A_1.$$

This enables us to write,

$$A_1 = \frac{A + iB}{2} \qquad A_2 = \frac{A - iB}{2}$$

Then the solution (12) can be written as,

$$x(t) = \exp(-\beta t)(A\cos(\omega' t) + B\sin(\omega' t)) \tag{14}$$

where $\omega' = \sqrt{\omega_0^2 - \beta^2}$. One can choose $B = 0$, by fixing appropriate initial conditions giving the following form of solution,

$$x(t) = A\exp(-\beta t)\cos(\omega' t) \tag{15}$$

Equation (15) shows a exponentially decaying term in conjunction with an oscillating terms with angular frequency $\omega' = \sqrt{\frac{k}{m} - \frac{r^2}{m^2}} = \sqrt{\omega_0^2 - \beta^2}$. ω_0 is the natural frequency of the system, that is angular frequency in absence of damping. More the damping longer the time period of oscillations.
Problem: Find out initial conditions which would lead to the equation (15).

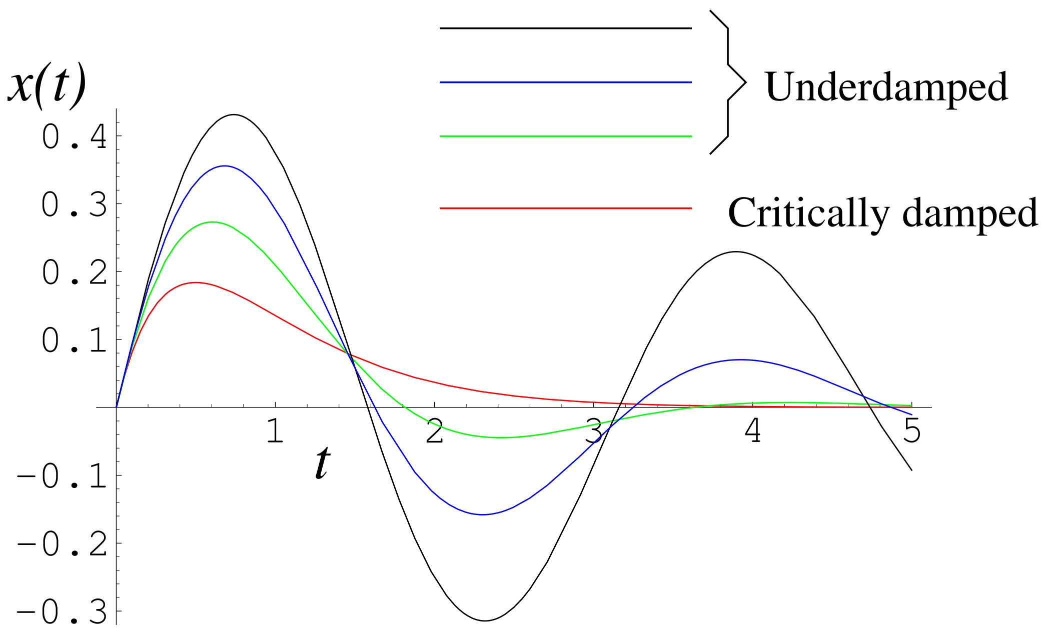

Figure 4: Underdamped oscillations for different resistances

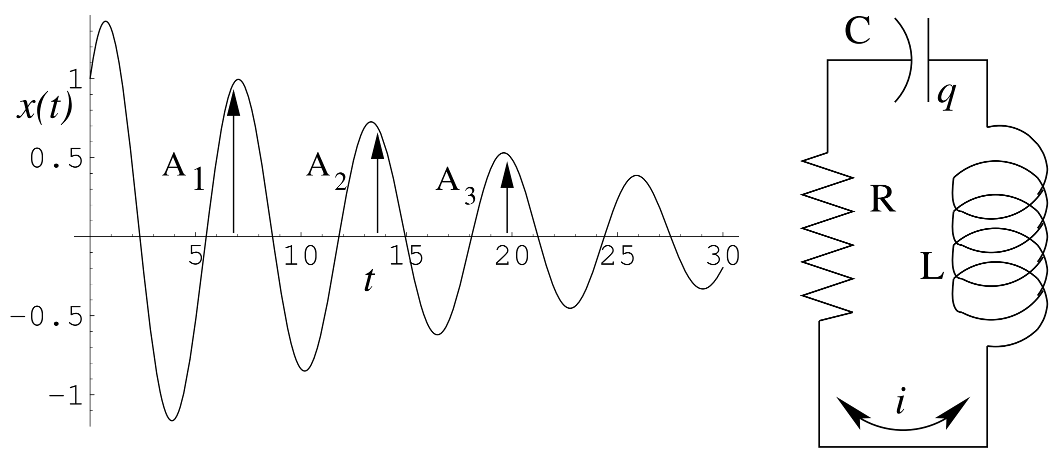

Figure 5: Underdamped case showing amplitudes and an L-C-R circuit

If one slowly increases the resistance a situation will come when the discriminant will vanish and the oscillations will cease and that happens for $\beta = \omega_0$(i.e. $r^2 = km$), which is the 'critical damping' condition discussed earlier.

Energy stored in a damped harmonic oscillator In the case of damped oscillator total energy of the system decreases with time. Since a damping factor $\exp(-\beta t)$ is present in the expression of displacement $x(t)$. he total energy is given by,

$$E(t) = \frac{1}{2}\exp(-2\beta t)kA^2, \tag{16}$$

where A is the initial amplitude.

Logarithmic decrement: Logarithmic decrement id a measure of amplitude decay and is given by,

$$\delta = \ln(A_n/A_{n+1}) = \beta T, \tag{17}$$

where A_n is the amplitude in the n^{th} period and T is the time period of oscillations.

Relaxation time or modulus of decay: The time taken for the amplitude to decay to $\frac{1}{e}$ of its original value.

$$\tau = \frac{1}{\beta} = \frac{m}{r}. \tag{18}$$

Quality factor: The quality factor measures the energy decay rate.

$$Q = \frac{\omega'}{2\beta} = \frac{\pi}{\delta} = \frac{\omega' m}{2r}. \tag{19}$$

That is the number of radians through which the damped system oscillates as its energy decays to $\frac{1}{e}$ of its original value. For small resistance $\omega' \approx \omega_0$, and $Q = \frac{\omega_0 m}{2r}$.
Problem: Show that $Q = 2\pi$(Energy stored in the system/Energy lost per cycle).
Problem: Draw the phase-space diagram of a damped simple harmonic motion.

Example: Damped SHM in LCR circuit: The voltage equation for the circuit is,

$$\begin{aligned} L\frac{di}{dt} + Ri + q/C &= 0 \\ \text{or,}\quad L\ddot{q} + R\dot{q} + q/C &= 0. \end{aligned} \tag{20}$$

where i is he current in the circuit. Comparing (20) with equation (1) we obtain the solution for charge on the capacitor as,

$$q(t) = q_{\pm}\exp(-Rt/2L \pm \sqrt{R^2/4L^2 - 1/LC}\ t) \tag{21}$$

Problem: Find out the constraints on the resistance, capacitance and inductance for different cases of damping.

Appendix:
Hyperbolic function:

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}$$

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2}$$

$$\cosh ix = \frac{\exp(ix) + \exp(-ix)}{2} = \cos x$$

$$\sinh ix = \frac{\exp(ix) - \exp(-ix)}{2} = i \sin x$$

$$\frac{d \cosh x}{dx} = \sinh x, \qquad \frac{d \sinh x}{dx} = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$